EBA 1180, lecture 21, 15 Nov 2022, Runar Ile

Plan 1. Repetition (problems from last week)

1d implicit differentiation

2 implicitly defined curves

5b The graphs of f, f', f''.

6b concave/convex functions

2. l'Hôpital's rule

1. Repetition

Probl 1d Implicit differentiation

$$x^3 - 3xy + y^2 = 0 \qquad (*)$$

We find an expression for y' in y and x by differentiating each side of $(*)$ with respect to x, and then solve the new equation for y'.

Partial calculations:

$$(xy)'_x \overset{\text{product rule}}{=} (x)'_x \cdot y + x \cdot y'_x$$

$$= 1 \cdot y + x \cdot y' = y + xy'$$

Chain rule: $(y^2)'_x = 2y \cdot y'$.

Then $(*)$ gives

$$3x^2 - 3(y + xy') + 2yy' = 0$$

that is $3x^2 - 3y + (2y - 3x)y' = 0$

$$(2y - 3x)y' = 3(y - x^2)$$

$$y' = \frac{3(y - x^2)}{2y - 3x} \qquad (**)$$

Assume $\underline{x = 2}$, we find the possible y-values by solving (*) with $x = 2$:

$$2^3 - 3\cdot 2\cdot y + y^2 = 0$$
$$y^2 - 6y = -8$$
$$(y-3)^2 = -8+9 = 1$$

so either $y - 3 = 1$ or $y - 3 = -1$

that is $\underline{\underline{y = 4}}$, $\underline{\underline{y = 2}}$

We use the point-slope formula to find the two tangent functions through the points $(2, 4)$ and $(2, 2)$.

$\underline{(2,4)}$ $\quad y' \overset{(**)}{=} \dfrac{3(4-2^2)}{2\cdot 4 - 3\cdot 2} = \dfrac{3\cdot 0}{2} = 0$

so the tangent function is constant: $\underline{\underline{h_1(x) = 4}}$

$(2,2)$ $\quad y' \overset{(**)}{=} \dfrac{3\cdot(2-2^2)}{2\cdot 2 - 3\cdot 2} = \dfrac{3\cdot(-2)}{-2} = 3$

so the point-slope formula gives

$$h_2(x) - 2 = 3(x-2)$$

that is $\underline{\underline{h_2(x) = 3x - 4}}$

Probl 2 Ellimination is the strategy.

In 1a, c and d we get two y-values for one x-value, so 1a, c and d cannot be the blue graph (to the right)

So 1b has to be the blue graph.

The red and the green graphs are symmetric and so their tangents are symmetric too. In particular the slopes are only changing signs (for a fixed x-value) This is the case for 1a and c. so

So 1d has to be the purple one (in the middle)

In 1a we have both y-values positive
In 1c one y-value is negative.
If the thicker horizontal lines are the x-axes, then

1a has to be the green (bottom) one

1c ——— " ——— red (upper left) one

<u>Probl 5b</u> In practice impossible to show directly that a graph is the derivative of another one.

<u>Strategy</u> Assume a graph is $f(x)$, and show that it is wrong. Repeat.

① I assume the blue (upper) one is $f(x)$.
(but I think it is not!)
Then the derivative $f'(x)$ equals the slope of the tangent to $f(x)$ which is (very) close to 0 for $5.5 \le x \le 6$. But neither of the two other graphs are (close to) zero in this interval.
Hence the assumption is wrong.
The blue is not $f(x)$.

② Assume the orange (middle) one is $f(x)$
(but I think it is not!)
Then $f'(x)$ has to be (very) close to 0 for $5 \le x \le 6$ (flat tangent). Then $f'(x)$ has to be the blue, and $f''(x) = [f'(x)]'$ is the olive one. But by the argument in ① this is impossible

③ Conclusion:

$f(x)$ has to be the remaining one: the olive (at the bottom)
$f''(x)$ has to be the blue one
and $f'(x)$ the orange one.

Start: 11.10

Probl 6b $\quad f(x) = \ln(x^2-2x+2) - \frac{x}{4} + 1$

Note: $x^2-2x+2 = (x-1)^2+1 \geq 1$

so $f(x)$ is defined on the whole number line.

$$f'(x) = \left[\ln(x^2-2x+2)\right]' - \frac{1}{4} + 0$$

Chain rule with

$u = x^2-2x+2$ and $g(u) = \ln(u)$

$u' = 2x-2$ $\qquad g'(u) = \frac{1}{u}$

$$= \frac{2x-2}{x^2-2x+2} - \frac{1}{4}$$

$$f''(x) = \frac{(2x-2)'(x^2-2x+2) - (2x-2)(x^2-2x+2)'}{(x^2-2x+2)^2}$$

$(2x)^2 - 2\cdot 2\cdot 2x + 2^2$

$$= \frac{2\cdot(x^2-2x+2) - \overbrace{(2x-2)(2x-2)}}{(x^2-2x+2)^2}$$

$$= \frac{2x^2-4x+4 - 4x^2+8x-4}{(x^2-2x+2)^2}$$

$$= \frac{-2x^2+4x}{(x^2-2x+2)^2} = \frac{-2x(x-2)}{\left[(x-1)^2+1\right]^2}$$

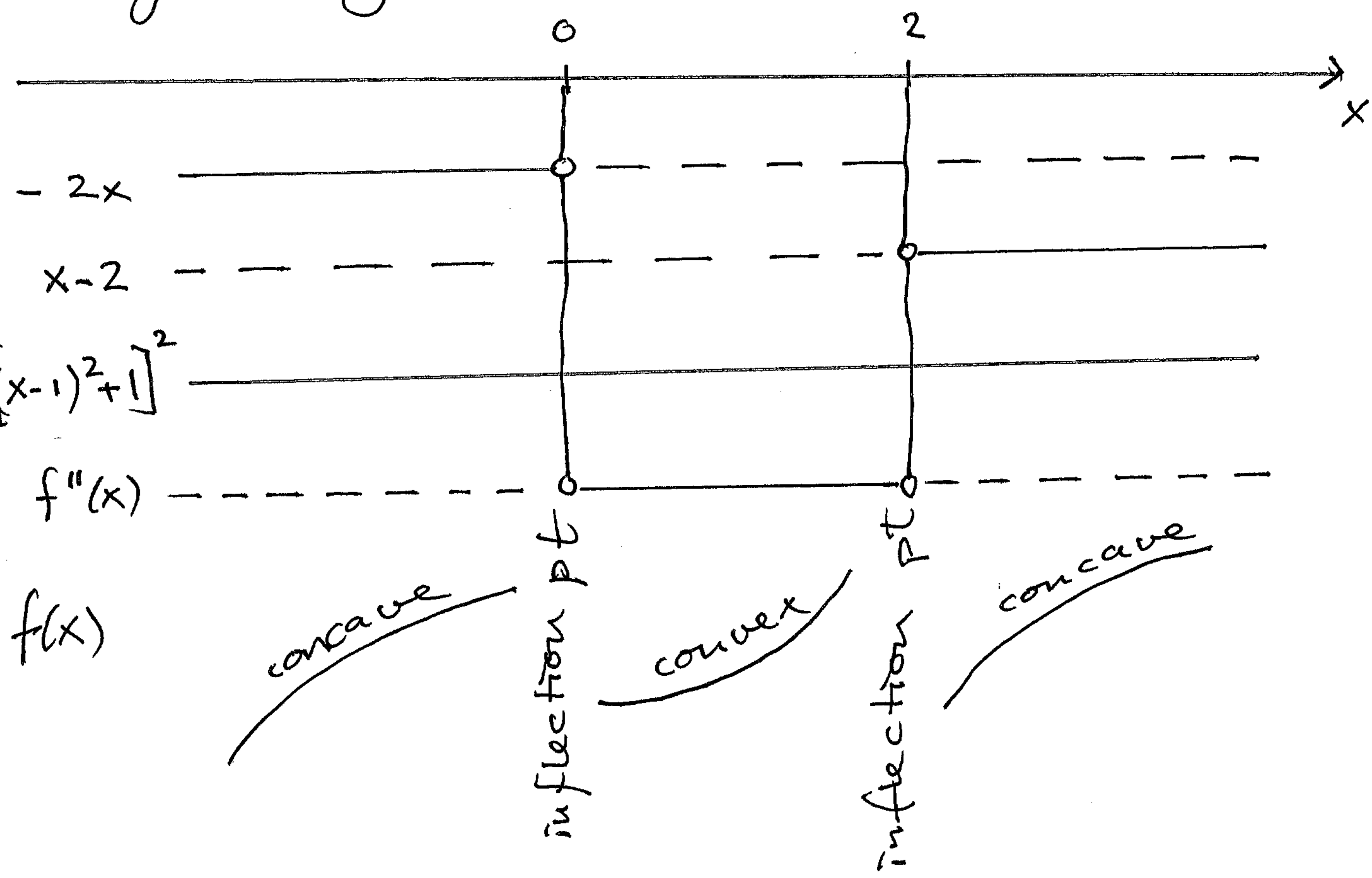
Solve equation $f''(x) = 0$

that is $-2x(x-2) = 0$

so $-2x = 0$ or $x - 2 = 0$

that is $\underline{\underline{x = 0}}$ or $\underline{\underline{x = 2}}$

Sign diagram for $f''(x)$.



<u>Conclusion</u>

$f(x)$ is concave for x in $\langle \infty, 0]$

$f(x)$ is convex for x in $[0, 2]$

$f(x)$ is concave for x in $[2, \infty\rangle$

Then the inflection points for $f(x)$ are

$\underline{\underline{x = 0}}$ and $\underline{\underline{x = 2}}$

2. l'Hôpital's rule

It is about limits of the type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Notation $\lim_{x \to 5} f(x)$ is the number which $f(x)$ is approaching when x is approaching 5.

Ex $f(x) = \frac{3x-3}{\ln(x)}$. Want to find $\lim_{x \to 1} f(x)$.

$$\left.\begin{array}{lll} \text{Numerator} & 3x-3 \xrightarrow[x \to 1]{} 3\cdot 1-3 = 0 \\ \text{Denominator} & \ln(x) \xrightarrow[x \to 1]{} \ln(1) = 0 \end{array}\right\} \frac{0}{0}\text{-express.}$$

Then we can use l'Hôpital's rule:

$$\lim_{x \to 1} f(x) \overset{l'Hôp}{=} \lim_{x \to 1} \frac{(3x-3)'}{[\ln(x)]'} = \lim_{x \to 1} \frac{3}{\frac{1}{x}} = \frac{3}{\frac{1}{1}} = \underline{\underline{3}}$$

check: $f(1.01) = \frac{3 \cdot 1.01 - 3}{\ln(1.01)} = 3.0150$

$f(0.99) = \frac{3 \cdot 0.99 - 3}{\ln(0.99)} = 2.9850$

Note Has to be $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Then differentiate numerator and denominator separately and try to find the limit of the new fraction.

$$\underline{\text{Ex}} \quad \lim_{x \to 0} \frac{3x}{e^x - 1} \qquad \left(\text{"}\frac{0}{0}\text{"} \right)$$

$$3x \xrightarrow[x \to 0]{} 0$$

$$e^x - 1 \xrightarrow[x \to 0]{} 1 - 1 = 0$$

$$\overset{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{3}{e^x} = \frac{3}{1} = \underline{\underline{3}}$$

E.g. $\frac{3 \cdot 0.01}{e^{0.01} - 1} = 2.9850$

$$\underline{\text{Ex}} \quad \lim_{x \to \infty} \frac{x^2}{e^x} \overset{\text{l'Hôp}}{=} \lim_{x \to \infty} \frac{2x}{e^x} \overset{\text{l'Hôp}}{=} \lim_{x \to \infty} \frac{2}{e^x} = \underline{\underline{0}}$$

"$\frac{\infty}{\infty}$"   "$\frac{\infty}{\infty}$"