EBA ~~2~~1180
Spring 23
Lecture 1

Integration & integration rules

- About me: Navn, bakgrunn osv.
- About the course:

Lectures:

Problem sessions:

→ Lecture plan, lecture notes, exercise sheets, messages & info: Itslearning.

Exams

→ Course paper EBA 29103
→ Final exam EBA 29104
+ retake exams

Topics

- Integration
- Matrix and vector computations
- Functions in two variables

& everything from last semester

Definite integrals

Bounds of integration

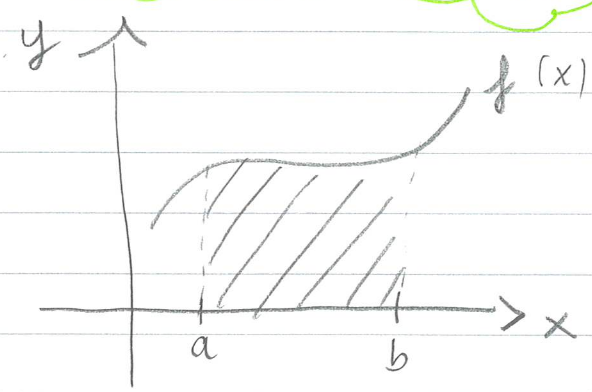
function we are integrating

Def: (Definite integral)

$$\int_a^b f(x)\,dx = \text{area of the region between the graph of } f \text{ and the x-axis in } [a, b]$$

integration sign

the integration variable



ASSUMPTIONS

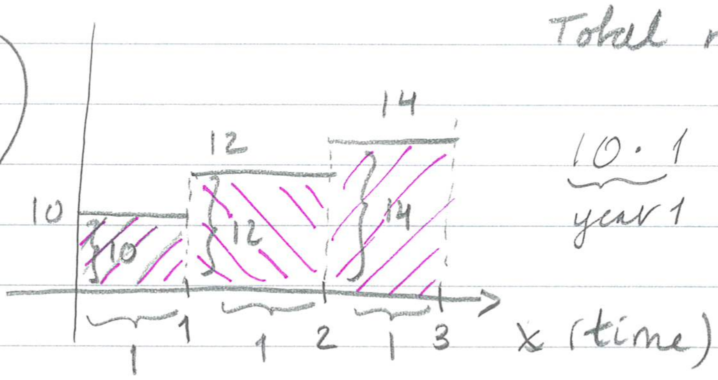
i) $f(x)$ is a continuous function on $[a, b]$.

ii) $f(x) \geq 0$ on $[a, b]$.

iii) $a < b$

Ex: Rental income over 3 years

Assume rent changed once per year;



Total rental income:

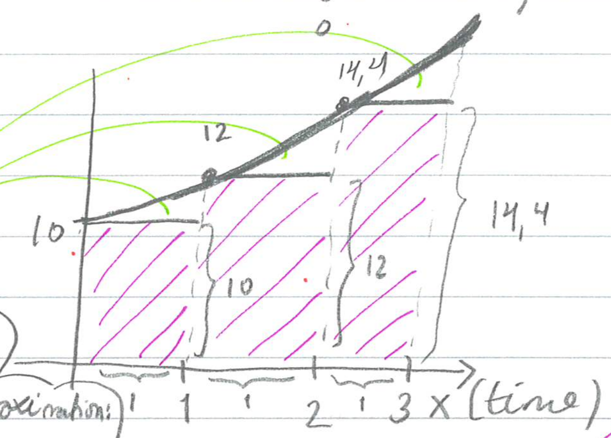
$$\underbrace{10 \cdot 1}_{\text{year 1}} + \underbrace{12 \cdot 1}_{\text{year 2}} + \underbrace{14 \cdot 1}_{\text{year 3}} = 36$$

Now: Assume we have a continously changing rent;

$$\underbrace{f(x)}_{\text{rent}} = 10 \cdot 1{,}2^{x}$$

Total rent income over 3 years

$= \int_0^3 10 \cdot 1{,}2^x$ = area under the graph between $[0,3]$



$f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$

$\approx 10 \cdot 1 + 12 \cdot 1 + 14{,}4 \cdot 1$

$= 36{,}4$

Riemann sum

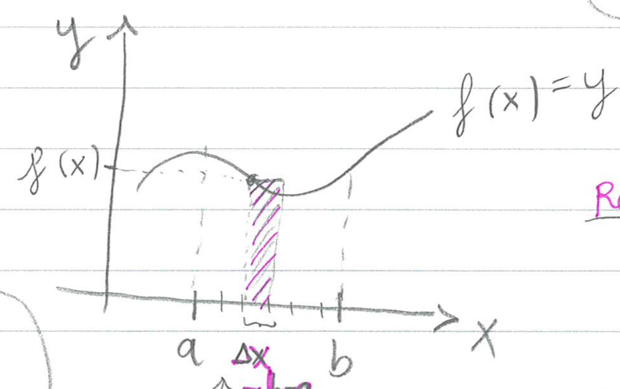
NOTE: This sum is an approximation: We are missing some bits of area which should have been included

How could we make the approximation better?

EXERCISE!

Idea: Make increments smaller

Note: In general



$\Delta x = \frac{b-a}{n}$

Cut up into n pieces

Rectangle has area:

$f(x)\Delta x \xrightarrow{\text{as } n \to \infty} f(x)dx$

NB: Lots of ways to make these shapes: choose left pt., right pt., upper value, lower value etc. For integrable func., these all become the same

(This is the main idea & intuition; How do we calculate these integrals?)

Anti-derivatives & indefinite integrals

Def: (Antiderivative)

An antiderivative of a function $f(x)$ is a function $F(x)$ s.t. $F'(x) = f(x)$.

So an antiderivative is a function s.t. if you differentiate it, you get the original function back

EX: $f(x) = 2x \Rightarrow$

$$F(x) = x^2$$

because

$$F'(x) = (x^2)' = 2x$$

Q: Can you think of another antiderivative of $f(x)$?

Say:

$F(x) = x^2 + 1$, because $F'(x) = (x^2+1)' = 2x$

Or more generally:

$$F(x) = x^2 + C, \text{ because } F'(x) = (x^2 + C)' = 2x$$

Fact: If $f(x)$ has an antiderivative $F(x)$, then any other antiderivative can be written $F(x) + C$ where C is a constant

Def: (Indefinite integral)

The indefinite integral of a function $f(x)$ is

$$\int f(x)\,dx = F(x) + C$$

all antiderivatives of $f(x)$

where $F'(x) = f(x)$.

The "basic" antiderivative of $2x$

Ex: $\int 2x\,dx = x^2 + C$

Recall previous example

a general constant, called the integration constant

Integration rules

$\boxed{\rightarrow \text{How can we compute } \int f(x)\,dx\,?}$

Ex: $\int 3 + x + x^2\,dx = 3x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$

integration constant

Check: Differentiate the answer:

$$\left(3x + \frac{1}{2}x^2 + \frac{1}{3}x^3\right)' = 3 + \frac{1}{2}\cdot 2x + \frac{1}{3}\cdot 3x^2$$

$$= 3 + x + x^2$$

Integration rules

To check that these are true: Differentiate RHS & see that you get what's inside the integral

i) Power rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

ii) $\int \frac{1}{x} dx = \ln|x| + C$

NOTE: Absolute value ⇝ ln only def. for pos. input

Integral of sum is sum of integrals:

iii) $\int u(x) + v(x)\, dx = \int u(x) dx + \int v(x) dx$

Constants can be moved outside the integral (multiplied)

multiplicative constants ...

iv) $\int c \cdot u(x) dx = c \int u(x) dx \quad$ (c is a constant)

v) Exponentials:

$$\int e^x dx = e^x + C$$

Because $(e^x)' = e^x$

$$\int a^x dx = \frac{1}{\ln(a)} a^x + C \qquad (a > 0)$$

Because $(a^x)' = a^x \ln(a)$

Note: Need to add integration constant C whenever solving an indefinite integral.

Ex: $\int 3x^5 dx = 3 \int x^5 dx = 3 \frac{x^6}{6} + C = \underline{\underline{\frac{1}{2} x^6 + C}}$

Ex: $\int 3x^5 + 6x^{12} dx \overset{iii)}{=} \int 3x^5 dx + \int 6x^{12} dx$

EXERCISE!

see prev. ex. (iv)

$= \frac{1}{2} x^6 + 6 \frac{x^{13}}{13} + C$

Even though we solve two indef. int. it's enough to add one int. const.

Can always say: $C_1 + C_2 := C$