EBA 1180
Spring 23

Lecture 26

Recap: • Antiderivative

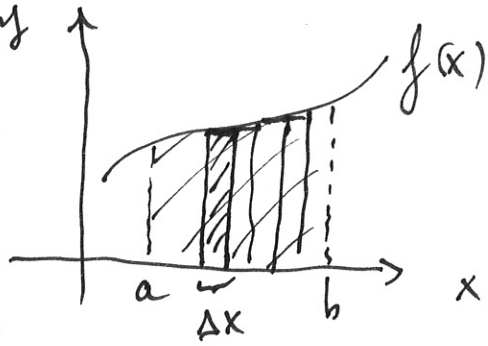
• $\int f(x)\,dx = F(x) + C$

don't forget the constant

Definite integrals:

• $\int_a^b f(x)\,dx$

Area under the graph of f and the x-axis



Ex: $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

TRICK!

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \dfrac{2}{3} x^{\frac{3}{2}} + C$

$= \dfrac{2}{3} x\sqrt{x} + C$

$x^{\frac{3}{2}} = x^1 x^{\frac{1}{2}} = x\sqrt{x}$

Ex: $\int \dfrac{1}{x^2}\,dx = \int x^{-2}\,dx = \dfrac{x^{-2+1}}{-2+1} + C$

$= \dfrac{x^{-1}}{-1} + C = -\dfrac{1}{x} + C$

Ex: $\int \frac{x^2-2x+3}{x}\,dx = \int \frac{x^2}{x} - \frac{2x}{x} + \frac{3}{x}\,dx$

$= \int x - 2 + \frac{3}{x}\,dx$

$= \int x\,dx - \int 2\,dx + \int \frac{3}{x}\,dx$

$= \underline{\underline{\frac{1}{2}x^2 - 2x + 3\ln|x| + C}}$

Substitution

" Reverse of the ~~quotient~~ Chain rule "

Ex: $\int e^{2x}\,dx = \int e^{u}\,dx = \int e^u \frac{1}{2}\,du$

Not same! Need du

$= \frac{1}{2}\int e^u\,du$

$= \frac{1}{2}e^u + C$

$u := 2x$

$du = u'\,dx$

u diff'ed wrt. x

$\frac{du}{dx} = u'(x)$

$du = 2\,dx$

$\boxed{\frac{1}{2}du = dx}$

$= \underline{\underline{\frac{1}{2}e^{2x} + C}}$

Differentiate this!

Check: $\frac{1}{\cancel{2}}e^{2x}\cdot\cancel{2} = e^{2x}$

FORMULA (Substitution):

$$du = u' \, dx$$

$$dx = \frac{1}{u'} \, du$$ where u' means the derivative of u wrt. x.

Ex: $\int x \sqrt{x^2+1} \, dx = \int \cancel{x} \sqrt{u} \frac{1}{2\cancel{x}} \, du$

TRY:
$u = x^2 + 1$
$du = \underbrace{2x}_{u'} \, dx$
$dx = \frac{1}{2x} \, du$

$$= \frac{1}{2} \int \sqrt{u} \, du$$

$$= \frac{1}{2} \int u^{\frac{1}{2}} \, du$$

$$= \frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{\cancel{2}} \frac{\cancel{2}}{3} u^{\frac{3}{2}} + C$$

$$= \frac{1}{3} u^{\frac{3}{2}} + C = \frac{1}{3} (x^2+1)^{\frac{3}{2}} + C$$

METHOD (substitution)

1) Find something ugly: An inner function? Do you see its derivative?

<u>If yes:</u> Substitution!! $u =$ ugly stuff

Ex: x^2+1 ; ugly inner func.

$2x$; derivative $\approx x \to$ factor in integrand.

2) Find dx via du from formula.

Ex: $dx = \frac{1}{2x} du$

3) Transform the integral to the form

$\int g(u)\,du$ no x's, just u's

Ex: $\int \frac{1}{2} \sqrt{u}\, du$

4) Solve this integral.

Ex: $\frac{1}{3} u^{\frac{3}{2}} + C$

5) Substitute back in for u, so the final answer is wrt. x.

Ex: $\frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

Ex: $\int \frac{\ln x}{x}\,dx = \int \frac{u}{\cancel{x}}\cancel{x}\,du$

$\ln x \cdot \frac{1}{x}$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$\boxed{dx = x\,du}$

$= \int u\,du$

$= \frac{1}{2}u^2 + C$

$= \frac{1}{2}(\ln x)^2 + C$

Ex: $\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\cancel{\sqrt{x}}}(-2\cancel{\sqrt{x}})\,du$

$u = 1-\sqrt{x}$
$= 1 - x^{\frac{1}{2}}$

$du = -\frac{1}{2}x^{-\frac{1}{2}}\,dx$

$= -\frac{1}{2\sqrt{x}}\,dx$

$dx = -2\sqrt{x}\,du$

$= -2\int e^u\,du$

$= -2e^u + C$

$= -2e^{1-\sqrt{x}} + C$

Ex: $\int e^{\sqrt{x}}\,dx = \int e^u\, 2\sqrt{x}\,du$

$u = \sqrt{x} = x^{\frac{1}{2}}$
$du = \frac{1}{2\sqrt{x}}\,dx$

$\boxed{dx = 2\sqrt{x}\,du}$

$= \int e^u\, 2u\,du$

$= 2\int \underbrace{e^u \cdot u}_{}\,du$

PRODUCT

Integration by parts

(Correspond to product rule for differentiation)

$$= 2\left(ue^u - e^u\right) + C$$

$$= 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

$$= \underline{\underline{2e^{\sqrt{x}}\left(\sqrt{x} - 1\right) + C}}$$

METHOD : Integration by parts

FORMULA : $\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' \, dx$

Functions of x

To compute the integral of a product:

1) See if one factor can be easily anti-diff'ed. $\boxed{u'}$
Ideally, the other factor is simple when its differentiated. $\boxed{v}$

Ex: $\int x \cdot e^x \, d\cancel{x}$

simple when differentiated; v

Doesn't get worse by anti-diff'ing: u'

2) Anti-differentiat u': Get u

Differentiate v: Get v'

Ex: $u' = e^x \Rightarrow u = e^x$

$v = x \Rightarrow v' = 1$

3) Use Formula:

Ex: $\int \underbrace{x}_{v}\, \underbrace{e^x}_{u'}\, dx = \underbrace{x \cdot e^x}_{v \cdot u} - \underbrace{\int e^x \cdot 1\, dx}_{\int u \cdot v'\, dx}$

$= xe^x - \underbrace{\int e^x dx}_{\text{Easy to compute!}}$

4) Compute the final integral:

Ex: $\int x e^x dx = \quad = xe^x - e^x + C$

Why does integration by parts hold?

$(u \cdot v)' = u' \cdot v + u \cdot v'$

Product rule for differentiation

$\Downarrow$ Integrate both sides of equality

$$u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

$\Downarrow$ Rearrange

$$\boxed{\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' \, dx}$$