EBA 1180
Lect. 28
(Spring 23)

<u>Warm up</u>: $\int 5x^2 e^x dx$

PLAN: $5\int \underbrace{x^2}_{v} \underbrace{e^x}_{u'} dx$

Integration by parts * 2

$$\int v \cdot u' dx = v \cdot u - \int v' \cdot u \, dx$$

Integration ~~by parts~~ of rational functions/fractions

Ex:

i) $\int \frac{2}{1-x} dx$ ii) $\int \frac{2x}{1-x^2} dx$ iii) $\int \frac{2}{1-x^2} dx$

<u>What to do?</u>

<u>Type i)</u>: $\int \frac{2}{1-x} dx = \int \frac{2}{u} (-1) du$

<u>substitution:</u>
$u = 1-x$
$du = (-1) dx$
$= -dx$

$$\boxed{dx = \frac{1}{\underbrace{-1}_{-1}} du = -du}$$

$$= -2 \int \frac{1}{u} du = -2 \ln|u| + C$$

$$= -2 \ln|1-x| + C$$

In general: If $a \neq 0$;

$$\int \frac{A}{ax+b}\,dx = \int \frac{A}{u}\,\frac{1}{a}\,du$$

Substitution:

$u = ax+b$
$du = a\,dx$ $\rightarrow$ $\boxed{dx = \frac{1}{a}\,du}$

$$= \frac{A}{a}\int \frac{1}{u}\,du$$

$$= \frac{A}{a}\ln|u| + C$$

$$= \frac{A}{a}\ln|ax+b| + C$$

FORMULA:

$$\boxed{\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C, \quad a \neq 0}$$

Ex (alternative):

$$\int \frac{x}{1-x}\,dx = \int -1 + \frac{1}{1-x}\,dx$$

Polynomial division if degree of numerator ≥ degree of denominator

$x : (-x+1) = -1$
$-(x-1)$
1 ← Remainder:

$$\frac{x}{1-x} = -1 + \frac{1}{1-x}$$

$$= \int -1\,dx + \int \frac{1}{1-x}\,dx$$

$$= -x - \ln|1-x| + C$$

From substitution

Type ii):

$$\int \frac{2x}{1-x^2}\,dx = \int \frac{\cancel{2x}}{u}\left(-\frac{1}{\cancel{2x}}\right)du$$

$u = 1-x^2$

$du = -2x\,dx$

$\boxed{dx = -\frac{1}{2x}\,du}$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|1-x^2| + C$$

Type iii):

$$\int \frac{2}{1-x^2}\,dx = \int \underbrace{\frac{\cancel{2}}{u}\,\frac{(-1)}{\cancel{2}x}}_{\text{Looks bad!}}\,du$$

Substitution:

$u = 1-x^2$

$du = -2x\,dx$

$\boxed{dx = \frac{-1}{2x}\,du}$

$x^2 = 1-u$

$x = \pm\sqrt{1-u}$

METHOD: Partial fractions → Delbrøks-oppspaltning

GOAL: Try to find A, B s.t.:

EX: $\displaystyle \underbrace{\frac{2}{1-x^2}}_{1-x^2=(1-x)(1+x)} = \frac{A}{1+x} + \frac{B}{1-x} \quad \Big| \cdot (1-x^2)$

$(x-2)(x+3)$

$$\cancel{2 = A(1-x) + B(1+x)}$$

$$2 = \frac{A}{\cancel{1+x}}(1-x)\cancel{(1+x)} + \frac{B}{\cancel{1-x}}\cancel{(1-x)}(1+x)$$

$$2 = A(1-x) + B(1+x)$$

$$2 = A - Ax + B + Bx$$

$$0 \cdot x + 2 = (B-A)x + (A+B)$$

Why does this hold?

$x=0$:

$2 = A+B$

$x=1$:

$B-A=0$

$\boxed{\text{Compare coefficients}}$

1) $B - A = 0 \Rightarrow B = A$

2) $A + B = 2$

Insert

$$A + A = 2$$

$$2A = 2$$

$$A = 1 \Rightarrow B = 1$$

Hence, $$\frac{2}{1-x^2} = \frac{1}{1+x} + \frac{1}{1-x}$$

iii) $$\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1+x} + \frac{1}{1-x}\,dx$$

Partial fractions

$$= \int \frac{1}{1+x}\,dx + \int \frac{1}{1-x}\,dx$$

$$= \ln|1+x| + (-\ln|1-x|) + C$$

+ signs

− signs

$$= \ln|1+x| - \ln|1-x| + C$$

$$\left(= \ln\frac{|1+x|}{|1-x|} + C \right)$$

Exercise sheet 27

2) MET 11803 Fall 2018

$$f(x) = \frac{e^{1-\sqrt{x}}}{\sqrt{x}}, \quad x > 0$$

a) $f'(x) = ?$ $\quad f'(x) = \dfrac{e^{1-\sqrt{x}}(-\sqrt{x}-1)}{2x\sqrt{x}}$

b) Show that f is decreasing in $D_f = (0,\infty)$.

c) Determine the limits $\lim\limits_{x\to 0^+} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.

d) Sketch the graph of f based on this.

a) $$f'(x) = \frac{\left(e^{1-\sqrt{x}}\right)'\sqrt{x} - e^{1-\sqrt{x}}\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}$$

$\sqrt{x} = x^{\frac{1}{2}}$

quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$$= \frac{\left(e^{1-\sqrt{x}}\right)'\sqrt{x} - \dfrac{e^{1-\sqrt{x}}}{2\sqrt{x}}}{x}$$

$$= \frac{\overbrace{e^{1-\sqrt{x}}}^{g'(u)}\overbrace{\left(-\dfrac{1}{2\sqrt{x}}\right)}^{u'}\sqrt{x} - \dfrac{e^{1-\sqrt{x}}}{2\sqrt{x}}}{x}$$

Chain rule:
$u = 1-\sqrt{x}$
$u' = -\frac{1}{2\sqrt{x}}$

$$= \frac{e^{1-\sqrt{x}}\left(-\sqrt{x} - 1\right)}{2x\sqrt{x}}$$

b) Suffices to show $f'(x) < 0$ for $x \in (0,\infty)$

element in

Look at f' from a):

Denominator: $\underbrace{\underbrace{x}_{+}\,\underbrace{\sqrt{x}}_{+}}_{+} > 0 \implies$ Positive denominator

Numerator:
- $e^u > 0$ for all u, in part. $e^{1-\sqrt{x}} > 0$ for all $x > 0$.
- $\underbrace{-\underbrace{\sqrt{x}}_{+}}_{-}\ \underbrace{-1}_{-} < 0$ for $x>0 \implies$ Negative numerator

$f'(x) < 0$ for $x > 0$. So f is decreasing in D_f.

6

c) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{1-\sqrt{x}}}{\sqrt{x}}$ $\longrightarrow e^{1-0} = e^1 = e$

$\longrightarrow$ "$\sqrt{0^+}$" $= 0^+$

$= \infty$

$\longrightarrow$ "$\frac{e}{0^+}$" $= +\infty$