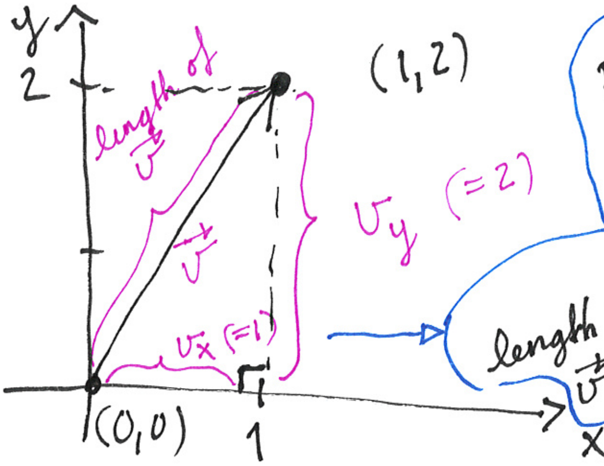
Geometric interpretation of vectors

EBA 1180 Spring 23

<u>Ex</u>: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

corresponds to an arrow from $(0,0)$ to (v_x, v_y):



<u>Pythagorean thm:</u>

$$v_x^2 + v_y^2 = (\text{length of } \vec{v})^2$$

$$\text{length of } \vec{v} = \sqrt{v_x^2 + v_y^2}$$

- A vector has a length (magnitude) and a direction.

$$\|\vec{v}\| = \left\| \begin{bmatrix} v_x \\ v_y \end{bmatrix} \right\| = \sqrt{v_x^2 + v_y^2}$$

<u>Ex ctd:</u> $\|\vec{v}\| = \sqrt{1^2 + 2^2} = \underline{\underline{\sqrt{5}}}$

<u>Def (Length of a vector):</u>

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

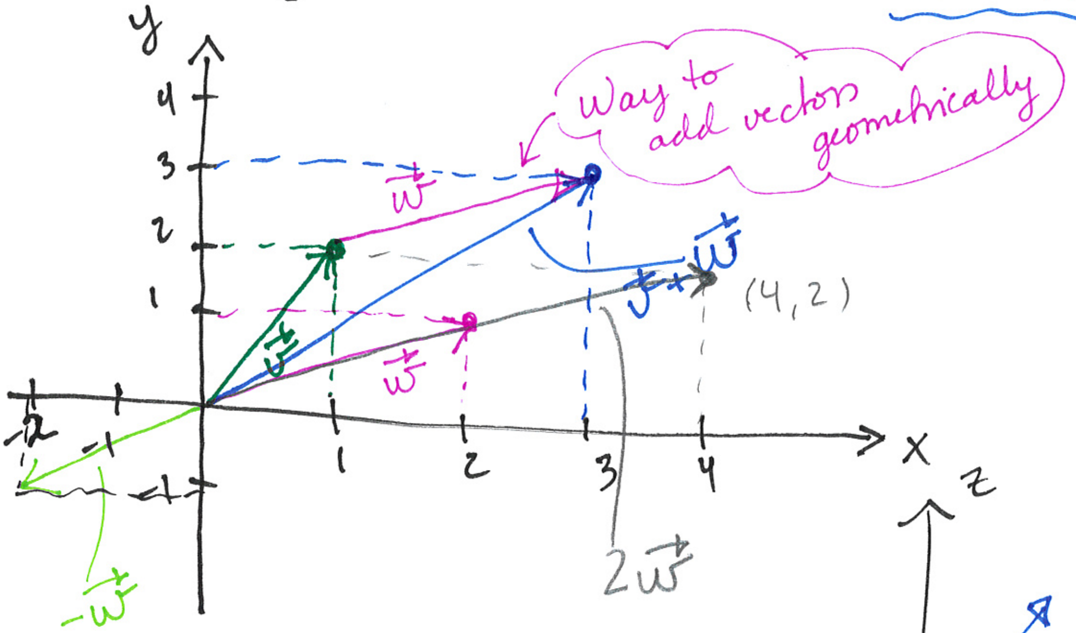
Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

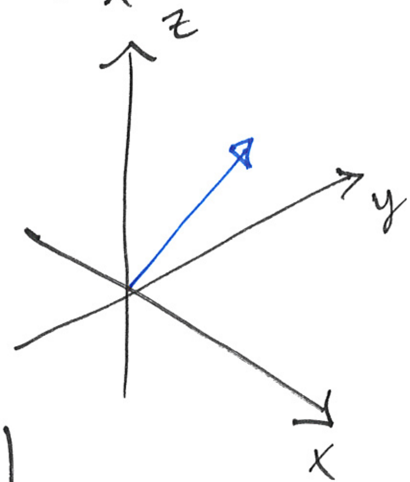
$$\vec{v} + \vec{w} = \begin{bmatrix} 1+2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$2\vec{w} = \begin{bmatrix} 2 \cdot 2 \\ 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$-\vec{w} = (-1) \cdot \vec{w} = \begin{bmatrix} (-1) \cdot 2 \\ (-1) \cdot 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

VISUALIZATION OF VECTOR OPERATIONS:





Determinants

$$A \longrightarrow \det(A) = |A|$$

$n \times n$ matrix (square)

determinant of A, a number

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\det(A) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

2×2

$= 2 \cdot 2 - 1 \cdot 1$

$= 3$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$

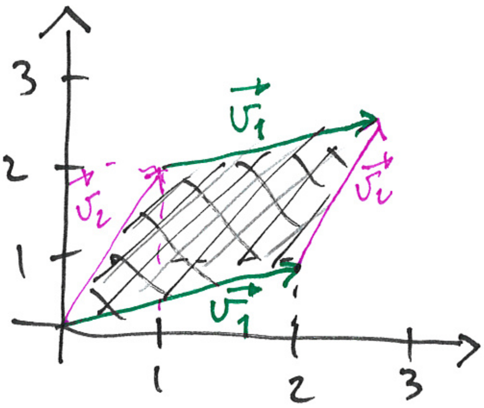
FORMULA: (Determinant, $n=2$)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Interpretation

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\vec{v}_1$ $\vec{v}_2$



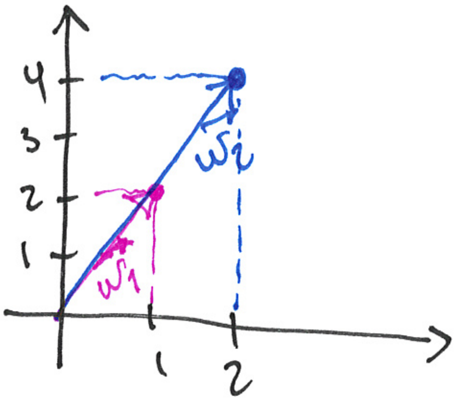
$|A| = 3$

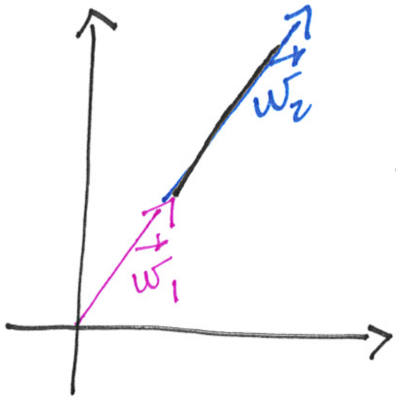
Can prove: $|\det(A)|$ = area of the parallellogram spanned by $\vec{v}_1$ and $\vec{v}_2$

$$\left| \vec{v}_2 \ \ \vec{v}_1 \right| = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 \cdot 1 - 2 \cdot 2 = -3$$

Ex: $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 2 = 0$

$\vec{w}_1$ = first column, $\vec{w}_2$ = second column



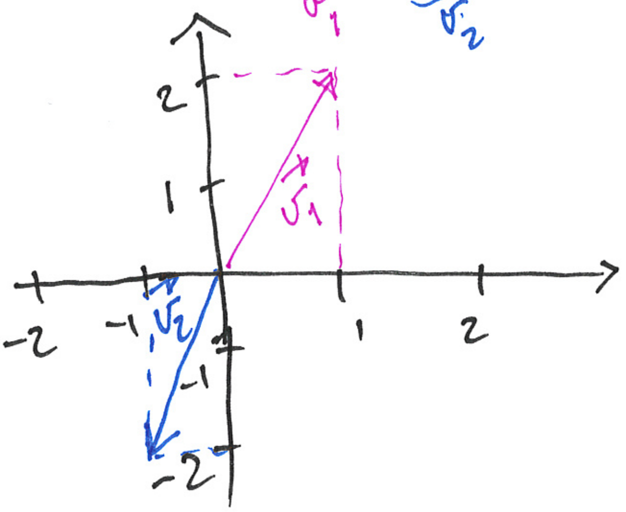


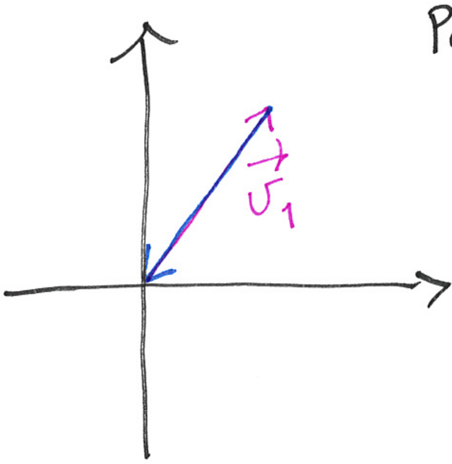
Parallellogram spanned by $\vec{w}_1$ and $\vec{w}_2$:

Area = 0

Ex: $\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = 1\cdot(-2) - (-1)\cdot 2 = 0$

$\vec{v}_1$ = first column, $\vec{v}_2$ = second column





Parallogram spanned by $\vec{v}_1$ and $\vec{v}_2$:

Area = 0

Result: (Determinant = 0, n = 2)

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \iff \vec{v}_1 = \begin{bmatrix} a \\ c \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$$

satisfy: One of the vectors is a scalar multiple of the other

Method for finding determinants:
Cofactor expansion

- The general case: $n \times n$ matrix

<u>Ex</u>: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 3 & 9 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$$

Cofactor expansion along row 1

where C_{11}, C_{12}, C_{13} are <u>cofactors</u>:

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

where M_{ij} is minor the determinant of the submatrix you get when you delete row i, column j.

$$= 1 \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 18 - 12 - (9 - 4) + 3 - 2 = \underline{\underline{2}}$$

NOTE: • Cofactor expansion along any row/column will give the same result.

• Any determinant can be computed via cofactor expansion.

TIP: Exploit the 0's!

Connection between the # solutions of linear systems & determinants

• If you start with $n \times n$ linear system (# eqns = # variables), then the corresp. coefficient matrix is an $n \times n$ matrix.

Ex:

$$\underbrace{\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}}_{3 \times 3 \text{ lin. system}}$$

Write this as: $A\vec{x} = \vec{b}$ (matrix form)

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 7 \\ 13 \end{bmatrix}$$

Coefficient matrix: 3×3 matrix

→ Since coefficient matrix of $n \times n$ lin. syst. is an $n \times n$ matrix, we can compute $|A|$:

Result: i) $|A| \neq 0 \Rightarrow$ One unique solution.

ii) $|A| = 0 \Rightarrow$ no solutions OR infinitely many solutions.

Theorem (Kramer's rule):

Consider a linear system, $A\vec{x} = \vec{b}$, with coefficient matrix A and r.h.s. $\vec{b}$, s.t. A is square $(n \times n)$ with $|A| \neq 0$. Then, the solution of the linear system is

$$x_1 = \frac{|A_1(\vec{b})|}{|A|}, \quad x_2 = \frac{|A_2(\vec{b})|}{|A|}, \quad \ldots,$$

$$x_n = \frac{|A_n(\vec{b})|}{|A|}$$

where $A_i(\vec{b})$ is the matrix you get when you replace the i'th column in A with $\vec{b}$.

Ex:

$$x + y = 4$$
$$x + ay = 6$$

x, y: variables (endogeneous)

a : parameter (exogeneous)