# Linear combinations

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Ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}$

three 3-vectors.

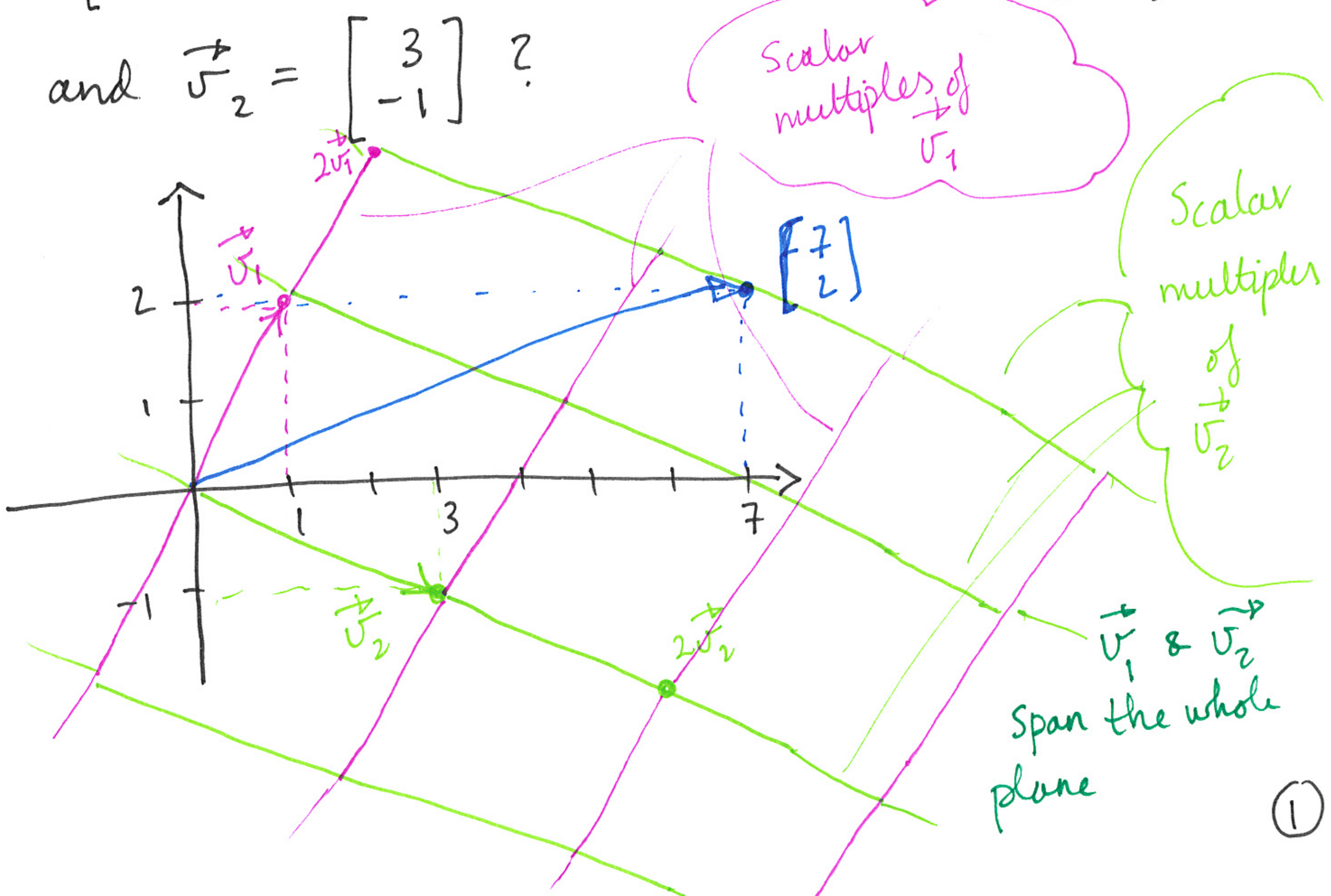
**Def:** (Linear combination)
A <u>linear combination</u> of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an expression of the form:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$$

where $c_1, c_2, c_3$ are given numbers.

In gen: $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ where $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$ are $m$-vectors

Ex: Is $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ a linear combination of $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$?



Want to find $c_1$ and $c_2$ s.t.

From def. of lin. comb.

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

a vector equation

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 3c_2 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + 3c_2 \\ 2c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \qquad \begin{cases} x + 3y = 7 \\ 2x - y = 2 \end{cases}$$

A $2 \times 2$ linear system:

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -12 \end{array}\right]$$

$(-2) \cdot$ row 1 add to row 2

$$\begin{aligned} c_1 + 3c_2 &= 7 \\ -7c_2 &= -12 \end{aligned} \quad \Rightarrow \quad c_2 = \frac{12}{7} \approx \underline{1{,}72}$$

$$c_1 = 7 - 3c_2 = \ldots = \frac{13}{7} \approx \underline{1{,}86}$$

So: $(c_1, c_2) \approx (1{,}86,\ 1.72)$

$$\left( 1{,}86 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1.72 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix} \right)$$

# Matrix multiplication

## Def: (Matrix multiplication)

$A, B$ matrices, and # columns in $A$ = # rows in $B$. Then,

$$A \cdot B$$

(matrix multiplication of $A$ by $B$)

is defined

Ex:

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 4 \\ 2 \cdot 3 + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$$

$$2 \times \underset{\substack{\text{\# cols.} \\ \text{in } A}}{2} = \underset{\substack{\text{\# rows} \\ \text{in } B}}{2} \times 1 = 2 \times 1$$

~~Matr~~ Matrix multiplication is defined!

Why defined like this?

$$x + 2y = 5$$
$$2x + y = 1$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A\vec{x} = \vec{b} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot x + 2 \cdot y \\ 2x + 1 \cdot y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$2 \times 2 \quad = \quad 2 \times 2 \rightsquigarrow 2 \times 2$

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot 2 & 1\cdot 2 + 2\cdot 1 \\ 2\cdot 1 + 1\cdot 2 & 2\cdot 2 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

Formula for $A\cdot B$: If $A\cdot B$ is defined with $A = [a_{ij}]$, $B = [b_{ij}]$, then $A\cdot B = C = [c_{ij}]$

where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$

NOTE: $AB \neq BA$

Even though $AB$ is defined, $BA$ may not be

Ex: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix}$

$2\times 2 \cdot 2\times 1 = 2\times 1$

$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$2\times 1 \cdot 2\times 2$, not defined: $1 \neq 2$

# Linear systems

Ex:
$$\begin{aligned} x + y + z + w &= 4 \\ x - y \quad + 2w &= 7 \\ 2x + 3y - z \quad &= 10 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$(3 \times 4 \cdot 4 \times 1 = 3 \times 1)$$

$$= \begin{bmatrix} x + y + z + w \\ x - y \quad + 2w \\ 2x + 3y - z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} = \vec{b}$$

That is, the lin. syst. can be written:

$$\boxed{A\vec{x} = \vec{b}}$$

The matrix form of the linear system

# Matrix algebra (computation with matrices)

## Operations:

1) Addition, subtraction: $A+B$, $A-B$ ($A, B$ same size)

2) Scalar multiplication: $c \cdot A$, $c$ is a number (always defined)

3) Matrix multiplication: $A \cdot B$ (when # cols. in $A$ = # rows in $B$)

4) Powers: $A^n$, (defined for $A$ square, $n = 1, 2, \ldots$)

Ex: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$, but

$$\underset{2 \times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}}^2 = \underset{2 \times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}} \cdot \underset{2 \times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}}$$

$3 \neq 2$; Not defined!

## Special matrices

→ The identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots,$$

$$I_n$$

→ We say that: $A^0 = I$

$\rightarrow$ I of suitable dimension

**Property:**
- $A \cdot I = A$ for any $A$
- $I \cdot A = A$

Ex:

$$\underset{2\times 2}{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \cdot \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} = \begin{bmatrix} 1\cdot1+2\cdot0 & 1\cdot0+2\cdot1 \\ 2\cdot1+1\cdot0 & 2\cdot0+1\cdot1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}$$

SIDE NOTE:

$$\underset{2\times 3}{A} \cdot \underset{3\times 3}{I_3}$$

$$\underset{2\times 2}{I_2} \cdot \underset{2\times 3}{A}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1\cdot1+0\cdot2 & 1\cdot2+0\cdot1 \\ 0\cdot1+1\cdot2 & 0\cdot2+1\cdot1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A}$$

# The transpose

$$A \longrightarrow A^T$$

$m \times n$ matrix $\qquad$ $n \times m$ matrix

( Read: "Transpose of A" OR "A transpose" )

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$

$2 \times 3 \qquad 3 \times 2$

("A flipped")

$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 1 & 7 & 3 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 7 \\ 0 & 4 & 3 \end{bmatrix}$

$3 \times 3 \qquad 3 \times 3$

Def: $A$ is a symmetric matrix if $A = A^T$.

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

"mirror"

so $A = A^T$, hence $A$ is symmetric.

$B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 4 \\ 0 & 4 & 7 \end{bmatrix}$,

$B = B^T$, hence $B$ is symmetric.

# Rules of matrix algebra:

**"Normal:"**

- $A + B = B + A$
- $A \cdot (B + C) = A \cdot B + A \cdot C$
- $A \cdot (BC) = (AB) \cdot C$

**NB:**

- $AB \neq BA$
- $(A+B)^2 = (A+B)(A+B)$

  $= A^2 + AB + BA + B^2$

  $\neq A^2 + 2AB + B^2$

## Determinants:

- $|A \cdot B| = |A| \cdot |B|$
- $|cA| = c^n |A|$, for $A$ $n \times n$
- $|A^T| = |A|$

## Transpose:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$

All can be proved, but none shall