Examples

EBA 1180
Spring 23

Ex:

3×3 linear system

$$\begin{cases} rx + 2y - z = 3 \\ x + (r+1)y - z = 3 \\ -x - 2y + rz = 1 - r \end{cases}$$

x, y, z: variables
r: parameter

$$|A| = \begin{vmatrix} r & 2 & -1 \\ 1 & r+1 & -1 \\ -1 & -2 & r \end{vmatrix} = r\begin{vmatrix} r+1 & -1 \\ -2 & r \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ -1 & r \end{vmatrix} - 1\begin{vmatrix} 1 & r+1 \\ -1 & -2 \end{vmatrix}$$

coefficient matrix

Signs of cofactor expansion 3×3:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= r\big((r+1)r - 2\big) - 2(r-1) - 1\big(-2 + (r+1)\big)$$

$$= r(r^2 + r - 2) - 2r + 2 + 2 - r - 1$$

$$= r(r+2)(r-1) - 3r + 3$$

$$= r(r+2)(r-1) - 3(r-1)$$

$$= (r-1)\big(r(r+2) - 3\big)$$

$$= (r-1)(r^2 + 2r - 3) = (r-1)(r-1)(r+3)$$

$$= (r-1)^2(r+3)$$

Quadratic formulla etc. to factorize

2 cases:

1) $|A| = 0$: $\quad (r-1)^2(r+3)=0$

Either no or ∞ many solutions

$r=1, \quad r=-3$

Which one?

i) $r=1$: Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 2 & -1 & 3 \\ -1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

(-1) · row 1 add row 2
Add row 1 to row 3

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Switch rows 2 & 3 for echelon form

Echelon form.

$0 = 3$ (never true) $\Rightarrow$

No solutions for $r=1$.

ii) $r=-3$:

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & 3 \\ 1 & -2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -3 & 2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right]$$

switch rows 1 and 2 – to simplify calculations

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & -4 & -4 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

3 * row 1 add to row 2
Add row 1 to row 3

(-1) · row 2 add to row 3

$\Rightarrow 0 = -5$ (never true)

$\Rightarrow$ No solutions for $r = -3$.

2) $|A| \neq 0$: $\quad (r-1)^2 (r+3) \neq 0$

$r \neq 1, \; r \neq -3 \Rightarrow$ Unique solution.

Find this via Cramer's rule: $\rightarrow$ Easier than Gaussian elimination due to parameters

To find x:

Know: $|A| = (r-1)^2 (r+3)$

Need: $|A_1(\vec{b})| = \begin{vmatrix} 3 & 2 & -1 \\ 3 & r+1 & -1 \\ 1-r & -2 & r \end{vmatrix}$

$\begin{bmatrix} 3 \\ 3 \\ 1-r \end{bmatrix}$

$$= 3 \begin{vmatrix} r+1 & -1 \\ -2 & r \end{vmatrix} - 2 \begin{vmatrix} 3 & -1 \\ 1+r & r \end{vmatrix} + (-1) \begin{vmatrix} 3 & r+1 \\ 1-r & -2 \end{vmatrix}$$

$$= \ldots = 2r^2 - r - 1$$

From Cramer's rule:

$$x = \frac{|A_1(\vec{b})|}{|A|} = \frac{2r^2 - r - 1}{(r-1)^2(r+3)}$$

See that 1 is a zero of numerator

$$= \frac{(2r+1)\cancel{(r-1)}}{(r-1)^{\cancel{2}}(r+3)} = \frac{2r+1}{(r-1)(r+3)},$$

$$r \neq 1, -3$$

- Can find y and z via similar calculations.

Vector equations

$$x \cdot \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + y \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ 0 \\ 4x \end{bmatrix} + \begin{bmatrix} 3y \\ -y \\ 2y \end{bmatrix} + \begin{bmatrix} 4z \\ -z \\ 6z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x + 3y + 4z \\ -y - z \\ 4x + 2y + 6z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x + 3y + 4z &= 3 \\ -y - z &= 1 \\ 4x + 2y + 6z &= 2 \end{aligned}$$

Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 4 & 2 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & -10 & -10 & -10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -20 \end{array}\right] \Rightarrow \quad 0 = -20$$

(never true)

$\Downarrow$

No solutions!

Inverse matrices

Def (Inverse matrix): Let A be an $n \times n$ matrix.

square

An inverse of A is a matrix A^{-1} s.t.

$$A \cdot A^{-1} = I \quad \text{and}$$

$$A^{-1} \cdot A = I.$$

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $|A| = 4 - 1 = 3 \neq 0$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \cdot \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 4-1 & -2+2 \\ 2-2 & -1+4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Can check:

$$\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Hence;

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{(from Def.)}$$

FORMULA (inverse, $n=2$):

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$|A| = ad - bc \neq 0$:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$|A| = 0 = ad - bc$:

No inverse of A.

FACTS : i) The inverse of A does not always exist.

Actually, A is invertible (i.e., A^{-1} exists) if and only if $|A| \neq 0$.

ii) If A has an inverse, it is unique.

iii) General formulas for A^{-1} when $|A| \neq 0$:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}^T$$

where c_{ij} are the cofactors.

Example:
$$2x + y = 4$$
$$x + 2y = 3$$

matrix form:
$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\vec{b}}$$

If A^{-1} exists:
$$\underbrace{A^{-1} A}_{I} \vec{x} = A^{-1} \vec{b}$$

Multiply with A^{-1} from left on both sides

$$I \vec{x} = A^{-1} \vec{b}$$
$$\vec{x} = A^{-1} \vec{b}$$

$$\vec{x} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 8 - 3 \\ -4 + 6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Prev. example

$$= \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \end{bmatrix}$$

So, $\underline{\underline{(x, y) = \left(\frac{5}{3}, \frac{2}{3}\right)}}$.

Advantage of A^{-1}: Quickly solve lin. syst. for many different r. h. s.