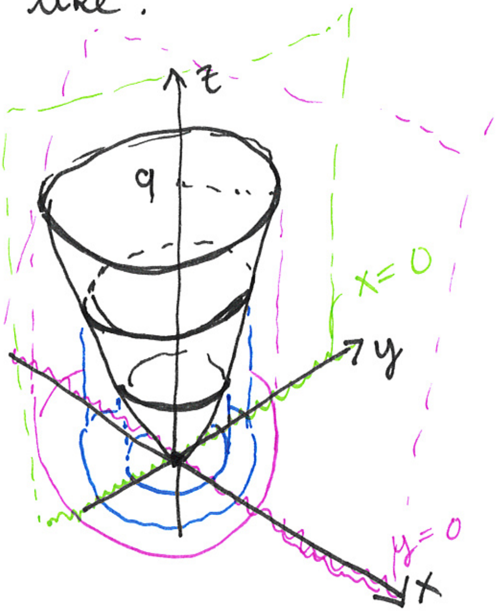
Recap: $f(x,y) = x^2 + y^2$

Level curve: $f(x,y) = c$

EBA 1180 Spring 23

Q1: Level curve for $c = 9$?

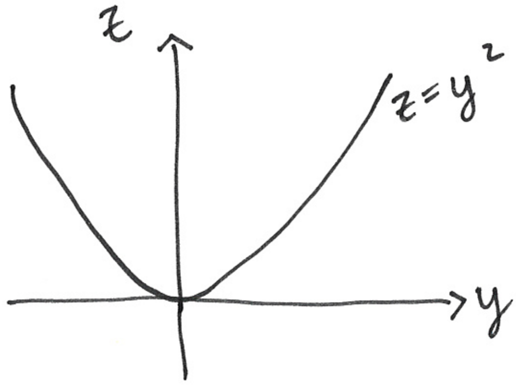
Q2: If $x = 0$, what does $z = f(x,y) = f(0,y)$ look like?



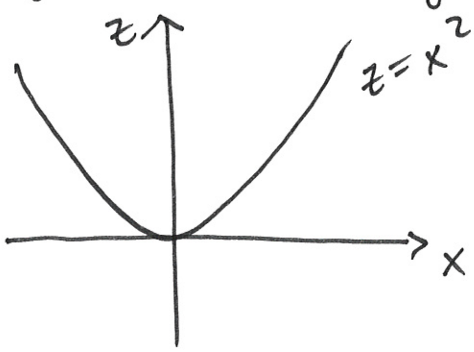
Q1: Circle, center $(0,0)$, $r = \sqrt{9} = 3$

More about graphical presentation of $f(x,y)$

Ex: Q_2: Cut $x = 0$: $z = f(0,y) = 0^2 + y^2 = y^2$



Cut: $y = 0$ $\quad z = f(x,0) = x^2 + 0^2 = x^2$



Linear functions

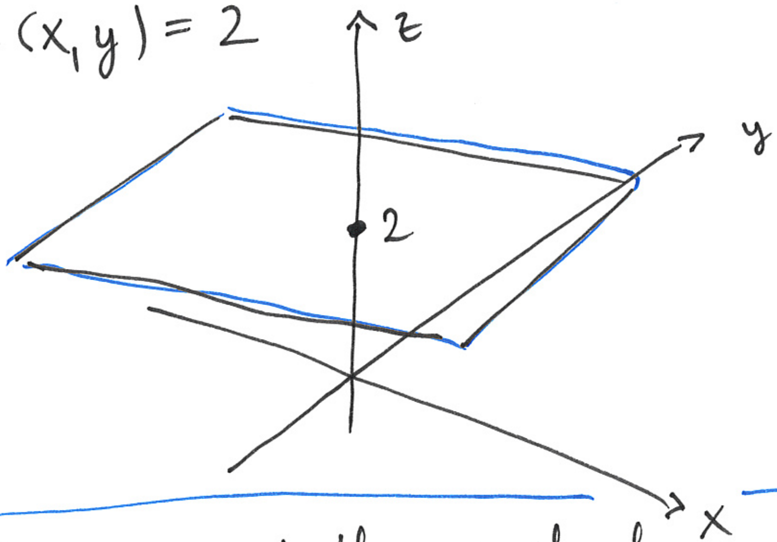
Def (Linear function): A function in two variables is linear if it can be written:

$$f(x,y) = ax + by + c$$

FACT: The graph of f is a plane $\iff$ f is linear.

Ex: $f(x,y) = 2$ → $a = b = 0$

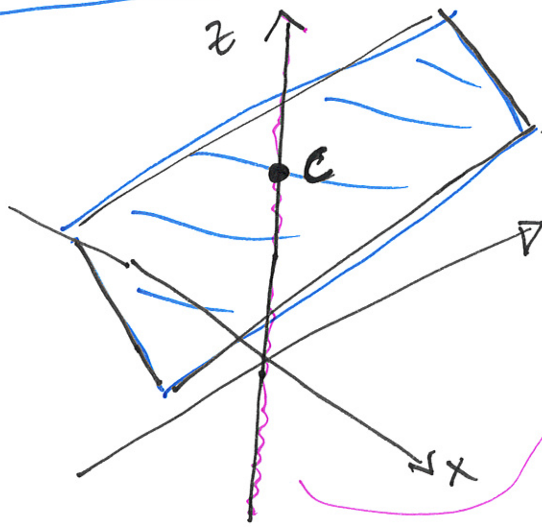
$z = f(x,y) = 2$



NOTE: The intersection of the graph of $f(x,y) = ax + by + c$ and the z-axis is $z = c$

Linear

$x = y = 0$ → $f(0,0) = c$



→ $f(x,y) = ax + by + c$

→ z varies, but $x, y = 0$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ Q: Which vectors $\vec{w}$ satisfy $\vec{v} \perp \vec{w}$?

Want to find $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ s.t. $\vec{v} \cdot \vec{w} = 0$:

$$\underset{\vec{v}}{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}} \cdot \underset{\vec{w}}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = 0$$

$$1 \cdot a + (-1)\, b + 2c = 0$$

$$a - b + 2c = 0$$

$$a = b - 2c$$

b, c free variables

$$\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b - 2c \\ b \\ c \end{bmatrix}$$

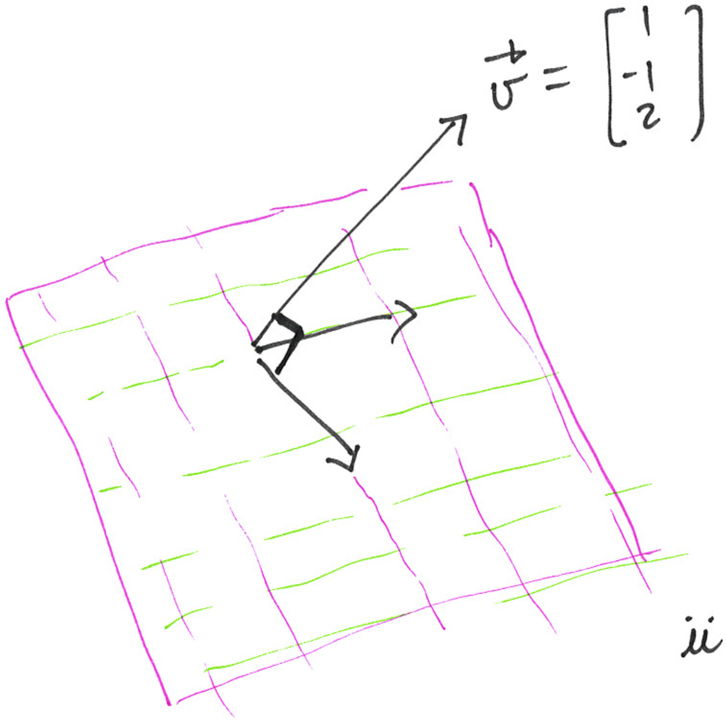
$$= \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} -2c \\ 0 \\ c \end{bmatrix}$$

$$= b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

All linear combinations of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$(c_1 \vec{v}_1 + c_2 \vec{v}_2)$

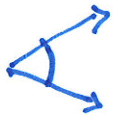
Conclusion: The vectors that are normal (90°) to $\vec{v}$ are all linear combinations of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.



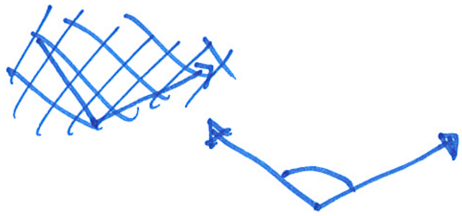
- <u>In general:</u>

i) $\vec{v} \cdot \vec{w} = 0 \iff \vec{v} \perp \vec{w}$

ii) $\vec{v} \cdot \vec{w} > 0 \iff$ angle $< 90°$



iii) $\vec{v} \cdot \vec{w} < 0 \iff$ angle $> 90°$



Linear functions with $c = 0$

$$f(x,y) = ax + by$$

$$z = ax + by$$

$$0 = ax + by - z \iff \begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Updownarrow$$

$$\begin{bmatrix} a \\ b \\ -1 \end{bmatrix} \perp \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Hence, the graph of $f(x,y) = ax + by$: All vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that are normal to $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$

90°

- This is a plane and $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ is its normal vector.

Ex: $f(x,y) = x - 2y$

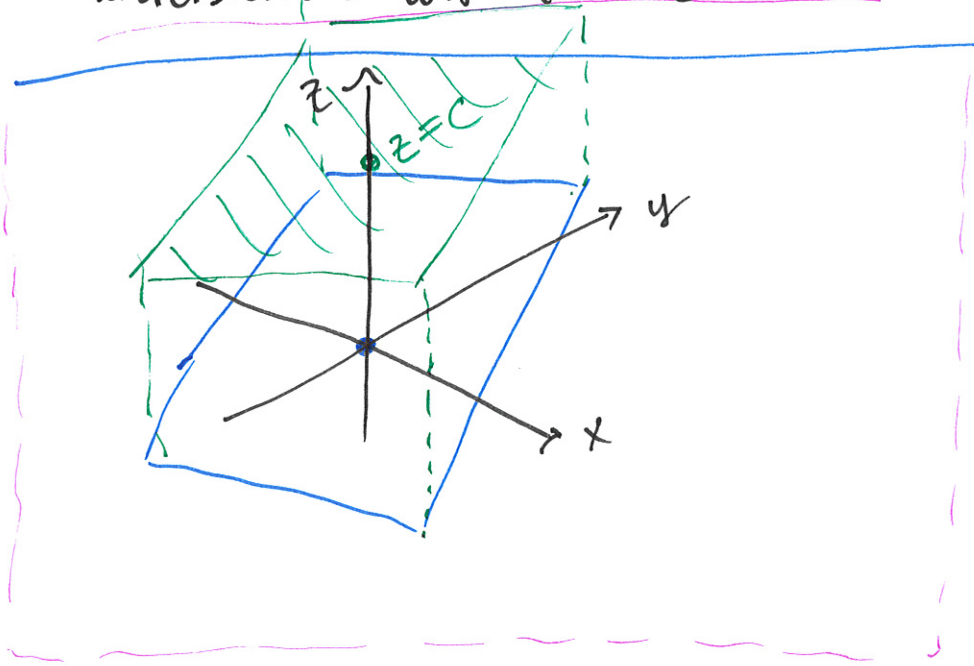
$$z = x - 2y \quad \Rightarrow$$
$$0 = x - 2y - z$$

Normal vector to the plane that is the graph of $f(x,y)$: $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

Conclusion: The graph of a linear function in two variables:

$$f(x,y) = ax + by + c$$

is a plane with normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ and intersection with the z-axis $z = c$.



Partial derivatives of functions of two variables

<u>Ex</u>: $f(x,y) = 3x + 4y - 5$

$f(x,y) = x^2 + y^2$

<u>Partial derivatives</u>: "is defined"

$$f'_x(x,y) := \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

READ: "Partial derivative of f w.r.t. x"

<u>TO COMPUTE</u>: Think of y as a constant. Use normal rules of differentiation to find f'_x.

$$f'_y(x,y) := \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

<u>COMPUTE</u>: Think of x as a constant.

<u>Ex</u>: i) $f(x,y) = 3x + 4y - 5$

$f'_x(x,y) = 3 + 0 - 0 = \underline{\underline{3}}$

$f'_y(x,y) = 0 + 4 - 0 = \underline{\underline{4}}$

ii) $f(x,y) = x^2 + y^2$

$f'_x(x,y) = 2x + 0 = \underline{2x}$

$f'_y(x,y) = 0 + 2y = \underline{2y}$

$f''_{xy}(x,y) = 0$

$f''_{xx}(x,y) = 2$

$f''_{yy}(x,y) = 2$, $f''_{yx} = 0$

Def: (Stationary point)
Let $f(x,y)$ be a function. A pt. $(x,y)=(a,b)$ is a stationary point for f if
$$f'_x(a,b) = 0 = f'_y(a,b)$$

- To find stationary points: Solve the system of eqns:

Solve for (x,y)
$$\begin{cases} f'_x(x,y) = 0 \\ f'_y(x,y) = 0 \end{cases}$$

The Hessian of $f(x,y)$:

Def (Hessian): The Hessian of $f(x,y)$ is the 2×2 matrix
$$\mathcal{H}(f)(x,y) = \begin{bmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{bmatrix}$$

Optimization: max/min

Def (Max/min):

i) (x^*,y^*) is a maximal pt./maximizer for f if $f(x^*,y^*) \geq f(x,y)$ for all $(x,y) \in D_f$.

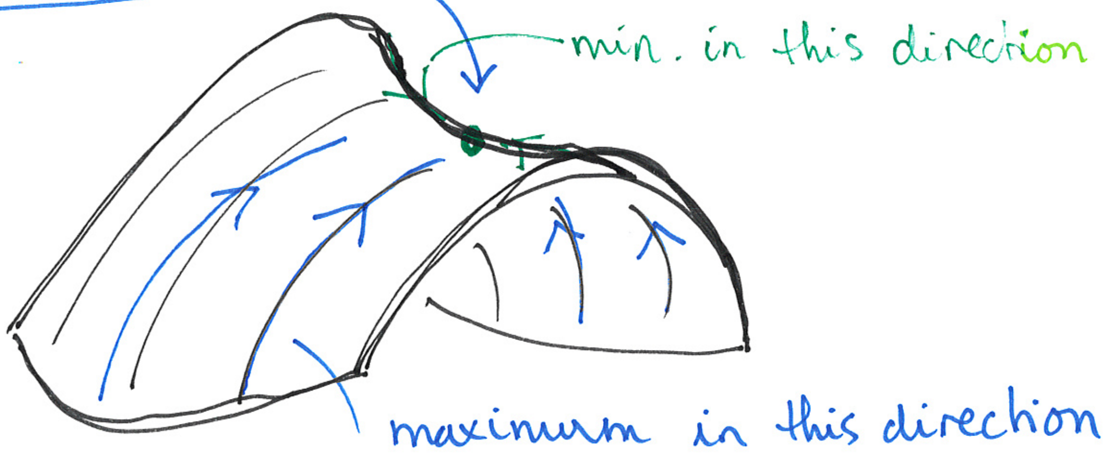
ii) (x^*,y^*) is a local max for f if $f(x^*,y^*) \geq f(x,y)$ for all (x,y) close to (x^*,y^*).

iii) (x^*, y^*) is a minimum pt. / minimizer for f if $f(x^*, y^*) \le f(x,y)$ for all (x,y) in D_f.

iv) (x^*, y^*) is a local min. for f if

$f(x^*, y^*) \le f(x,y)$ for all (x,y) close to (x^*, y^*).

v) A stationary point (x^*, y^*) of f that is neither a local max nor local min is called a saddle point.



max = global max
min = global min

KEY RESULT: If (x^*, y^*) is a max/min for f, then we have either:

Candidate points

i) (x^*, y^*) is a stationary point for f (i.e., $f'_x = f'_y = 0$ at (x^*, y^*))

ii) Either f'_x or f'_y is not defined at (x^*, y^*).

critical points

iii) (x^*, y^*) is a boundary point of D_f.

The second derivative test

Result: (The second derivative test)

If (x^*, y^*) is a stationary point of f, we compute

$$H(f)(x^*,y^*) = \begin{bmatrix} f''_{xx}(x^*,y^*) & f''_{xy}(x^*,y^*) \\ f''_{yx}(x^*,y^*) & f''_{yy}(x^*,y^*) \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

We have that;

1) If $\underbrace{\det H(f)(x^*,y^*)}_{AC-B^2} > 0$ and $\underbrace{\operatorname{tr} H(f)(x^*,y^*)}_{A+C} > 0$, then (x^*, y^*) is a local min.

"trace"

2) If $\det H(f)(x^*,y^*) > 0$ and $\operatorname{tr} H(f)(x^*,y^*) < 0$, then (x^*,y^*) is a local max.

3) If $\det H(f)(x^*,y^*) < 0$, then (x^*,y^*) is a saddle point.

<u>NOTE</u>: If $\det \mathcal{H}(f)(x^*, y^*) = 0$, the test is inconclusive.