EBA1180 Spring 23

Recap: $f(x,y) = 2x^2 + xy$

Q1: Stationary points?

$f'_x = 0$
$f'_y = 0$

Q2: Classify?

Q1: $f'_x = 4x + y \overset{!}{=} 0 \Rightarrow y = 0$

$f'_y = x \overset{!}{=} 0$

One stationary point: $(x,y) = (0,0)$

Q2: $$H(f) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$$

$\det H(f) = 0 - 1 = -1 < 0 \Rightarrow$ Saddle-pt. !

Stationary points ctd.

Ex: $f(x,y) = x^3 + 3xy + y^3$, $D_f = \mathbb{R}^2$

no boundary pts.

Partial derivatives exist everywhere: No pts. where either f'_x or f'_y are not def.

Stationary points (also the candidate points):

$$f'_x = 3x^2 + 3y = 0$$
$$f'_y = 3x + 3y^2 = 0$$

First order conditions (FOC)

$$\begin{cases} x^2 + y = 0 & \Rightarrow y = -x^2 \quad (\star) \\ x + y^2 = 0 & \rightarrow x + (-x^2)^2 = 0 \end{cases}$$

$$x + x^4 = 0$$

$$x \cdot (1 + x^3) = 0$$

$x = \underline{0}$

$(\star)$: $y = -0^2 = \underline{0}$

$1 + x^3 = 0$

$x^3 = -1$

$x = \sqrt[3]{-1} = \underline{-1}$

$(\star)$: $y = -(-1)^2 = \underline{-1}$

So the stationary points are $(0,0)$ and $(-1,-1)$.
Since there are no type ii) or type iii) points, these are also the candidate points.

The second derivative test

Ex ctd. $f(x,y) = x^3 + 3xy + y^3$

$f'_x = 3x^2 + 3y$

$f'_y = 3x + 3y^2$

Candidate points:

$(x^*, y^*) = (0,0), (-1,-1)$

$f(0,0) = 0^3 + 3 \cdot 0 \cdot 0 + 0^3 = \underline{0}$

$f(-1,-1) = (-1)^3 + 3 \cdot (-1) \cdot (-1) + (-1)^3 = -1 + 3 - 1 = \underline{1}$

Classify the candidate points using the second derivative test :

Hessian: $$H(f) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} 6x & 3 \\ 3 & 6y \end{bmatrix}$$

Candidate point (0,0):

$$H(f)(0,0) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \Rightarrow \det H(f)(0,0) = 0\cdot 0 - 3\cdot 3 = -9 < 0$$

$\Rightarrow$ (0,0) is a saddle point for f.

Second derivative test : 3)

Candidate point (−1,−1):

Hessian: $$H(f)(-1,-1) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$$

$$\det H(f)(-1,-1) = 36 - 9 = 27 > 0$$

since > 0, we also need to compute trace to classify

$$\operatorname{tr} H(f)(-1,-1) = -6 + (-6) = -12 < 0$$

$\Downarrow$ Second derivative test: 2)

(−1,−1) is a local max. for f.

Global max/min. ?

Conclusion: $f(x,y) = x^3 + 3xy + y^3$ has no minimum. —— " —— has a local maximum $f(-1,-1) = 1$.

NOTE: $f(10,10) = 10^3 + 3\cdot 10\cdot 10 + 10^3$

$= 2300 > 1 = f(-1,-1)$; value in local max

$\Rightarrow (-1,-1)$ is not a global max. for f

$\Rightarrow f$ has no global max.

Tangents of level curves

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

Level curve: All (x,y) s.t. $f(x,y) = c$.

$$x^2 - 2x + y^2 + 4y = c$$

TRICK: Complete the squares:

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = c + 1 + 4$$

$$(x-1)^2 + (y+2)^2 = c + 5 \quad (*)$$

$\underline{c+5>0}$:

So $c > -5$

$\Rightarrow$ Level curve is a circle with center in $(1,-2)$ and $r = \sqrt{c+5}$

$\underline{c+5=0}$: I.e. $c=-5 \Rightarrow$

Level curve is $(x,y) = (1,-2)$

(because sum of squares is 0 $\Rightarrow$ squares must be $0 \Rightarrow x-1=0$ and $y+2=0$)

$\underline{c+5<0}$:

I.e. $c<-5 \Rightarrow$

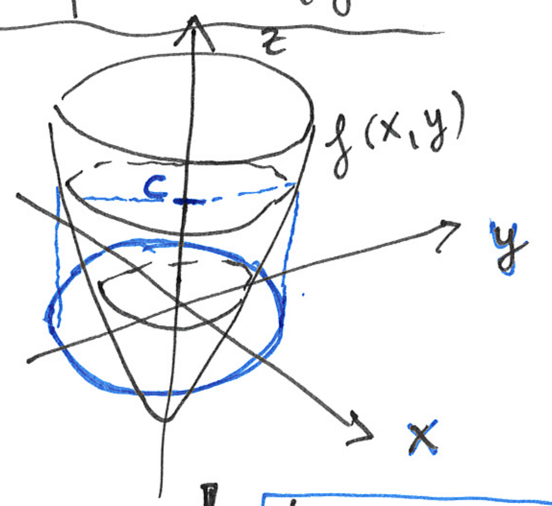
($\star$) is never true

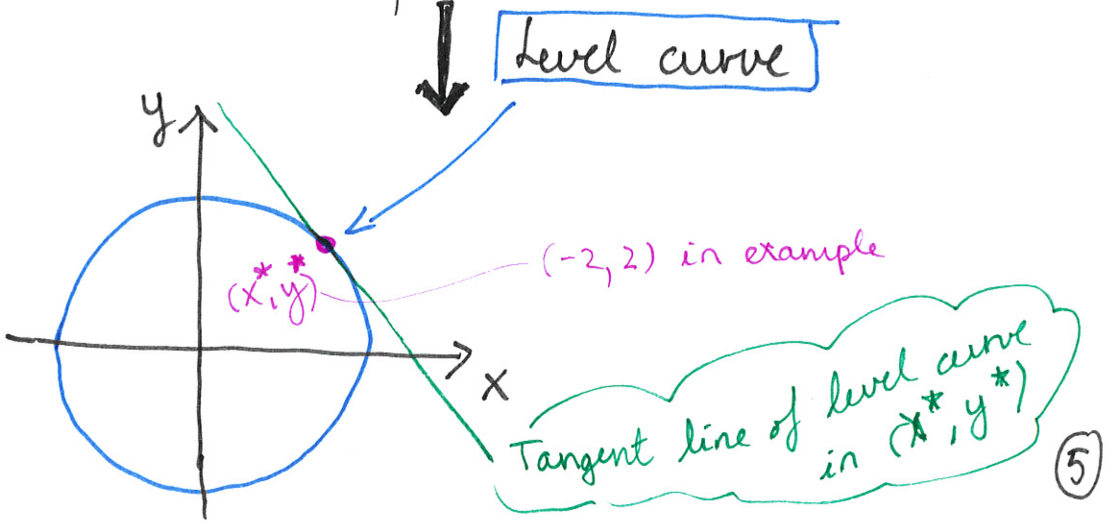
$\Rightarrow$ No level curve.

(i.e. f never takes a value $c<-5$)

How to find the tangent line of a level curve in a point (x,y)

Where are we?





Ex ctd: Tangent line of level curve of f in $(x, y) = (-2, 2)$?

$$f(x,y) = x^2 - 2x + y^2 + 4y$$

1. Find level of level curve corresponding to the point:

$$z = f(-2, 2) = (-2)^2 - 2\cdot(-2) + 2^2 + 4\cdot 2$$

$$= 20 \Rightarrow c = 20$$

So: Level curve at level 20 corresponds to $(x, y) = (-2, 2)$.

2. Point-slope formula:

slope: unknown

$$y - 2 = k\,(x - (-2))$$

$$y = k\,(x+2) + 2 \qquad (\sim)$$

What is the slope k? 3. Implicit differentiation

Found via implicit differentiation: Think $y = y(x)$:

$$(\underbrace{x^2 - 2x + y^2 + 4y}_{f(x,y)})'_x = (\underbrace{20}_{c})'_x \quad ; \text{ Level curve}$$

$$2x - 2 + 2\underbrace{y(x)\,y'(x)}_{\text{chain rule}} + 4y'(x) = 0$$

$$2x - 2 + 2yy' + 4y' = 0$$

Solve for y':

$$\underbrace{y'}_{} = -\frac{2x-2}{2y+4} = -\frac{f'_x}{f'_y}$$

OBSERVATION

$\frac{dy}{dx}$; the slope of a lin. func.

$y = ax + b$

$y' = a$; the slope

4. Insert $(x, y) = (-2, 2)$:

$$y'\big|_{(-2,2)} = -\frac{2\cdot(-2)-2}{2\cdot 2+4} = \underbrace{\frac{3}{4}}_{}$$

The slope of the tangent line in $(-2, 2)$

Hence, from $(\sim)$:

$$y = \frac{3}{4}(x+2) + 2$$

$$= \frac{3}{4}x + \frac{3}{2} + 2 = \frac{3}{4}x + \frac{7}{2}$$

HOLDS IN GENERAL

Result: If $f(x, y) = c$, then

$$f'_x + f'_y\, y' = 0$$

Implicit differentiation + chain rule (same arg. as above)

Hence,

$$\underbrace{y'}_{\text{slope}} = -\frac{f'_x}{f'_y}$$