EBA1180
Spring 23

Recap: $f(x,y) = \sqrt{x^2+y^2}$

let $u = x^2+y^2$

$$f'_x(x,y) = (\sqrt{u})'_x = (u^{\frac{1}{2}})'_x$$

$f'_y(x,y) =$

$$= \frac{1}{2}u^{-\frac{1}{2}} \cdot u'_x$$

CHAIN RULE

$$= \frac{1}{2\sqrt{u}}\, 2x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

By symmetry OR same arg. again:

$$f'_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$$

oo

Interpretation of partial derivatives

What does $f'_x(a,b)$ and $f'_y(a,b)$ mean?

Ex: $f(x,y) = x^3 - 3xy + y^3$

$$\Rightarrow f'_x(x,y) = 3x^2 - 3y, \quad f'_y(x,y) = -3x + 3y^2$$

OBS! minus

Let $(x,y) = (2,1)$. Then,

$$f(2,1) = 2^3 - 3\cdot 2\cdot 1 + 1^3 = 3$$

$$f'_x(2,1) = 3\cdot 2^2 - 3\cdot 1 = \underline{9}$$

SAME

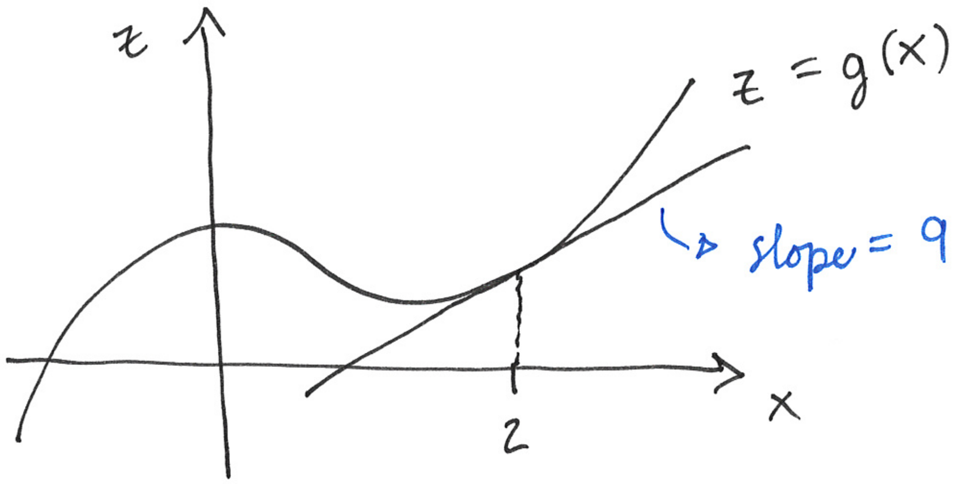
$$f'_y(2,1) = -3\cdot 2 + 3\cdot 1^2 = \underline{-3}$$

In the x-direction $(y=1)$:

$$f(x,1) = x^3 - 3x\cdot 1 + 1^3 = x^3 - 3x + 1 =: g(x)$$

Defined as

$$\underbrace{g'(x)}_{=f'_x(x,1)} = 3x^2 - 3 \quad , \quad \underbrace{g'(2)}_{=f'_x(2,1)} = 3\cdot 2^2 - 3 = \underline{9}$$



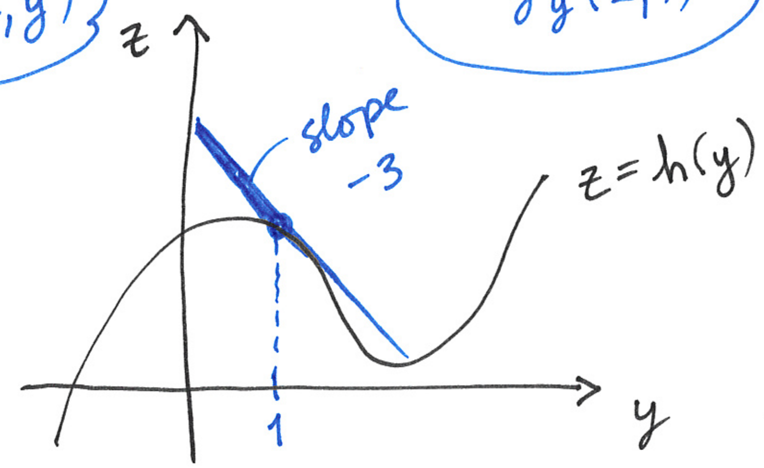
In the y-direction $(x=2)$:

$$f(2,y) = 2^3 - 3\cdot 2\cdot y + y^3 = 8 - 6y + y^3 =: h(y)$$

Defined as

$$\Rightarrow \underbrace{h'(y)}_{=f'_y(2,y)} = -6 + 3y^2, \text{ so } \underbrace{h'(1)}_{=f'_y(2,1)} = -6 + 3 = \underline{-3}$$

same



Linear approximation of $f(x,y)$ at (x_0, y_0): → Tangent plane of f at (x_0, y_0)

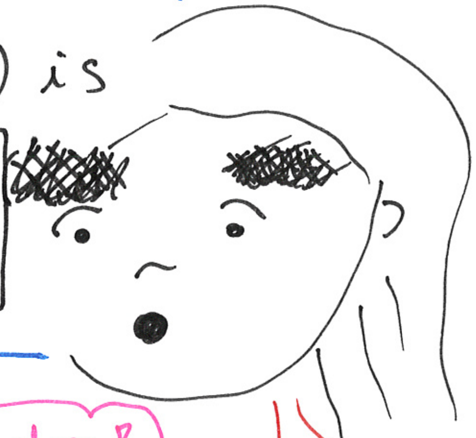
$$L(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

The gradient

Def (Gradient):

The gradient of $f(x,y)$ is

$$\underbrace{\nabla f}_{\text{"The gradient of } f\text{"}} = \underbrace{\begin{bmatrix} f'_x \\ f'_y \end{bmatrix}}_{\text{A vector!}}$$



Ex: • $f(x,y) = x^2 - 2x + y^2 + 4y$

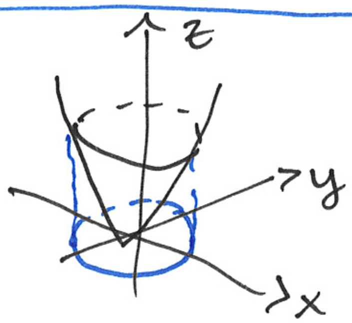
$$\nabla f = \begin{bmatrix} 2x-2 \\ 2y+4 \end{bmatrix}$$

• The gradient of f in $(x,y) = (-2,2)$:

$$\nabla f(-2,2) = \begin{bmatrix} 2\cdot(-2)-2 \\ 2\cdot 2+4 \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$$

NOTE: The gradient at a point is a normal vector to the tangent line of the level curve at that point.

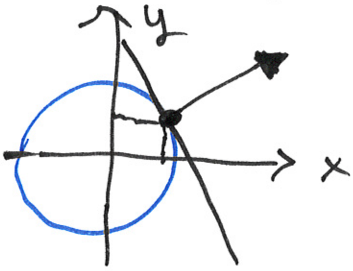
Ex: $\nabla f(-2,2) = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$



Tangent line of level curve at $(-2,2)$:

$$y = \frac{3}{4}x + \frac{7}{2}$$

Previous lecture

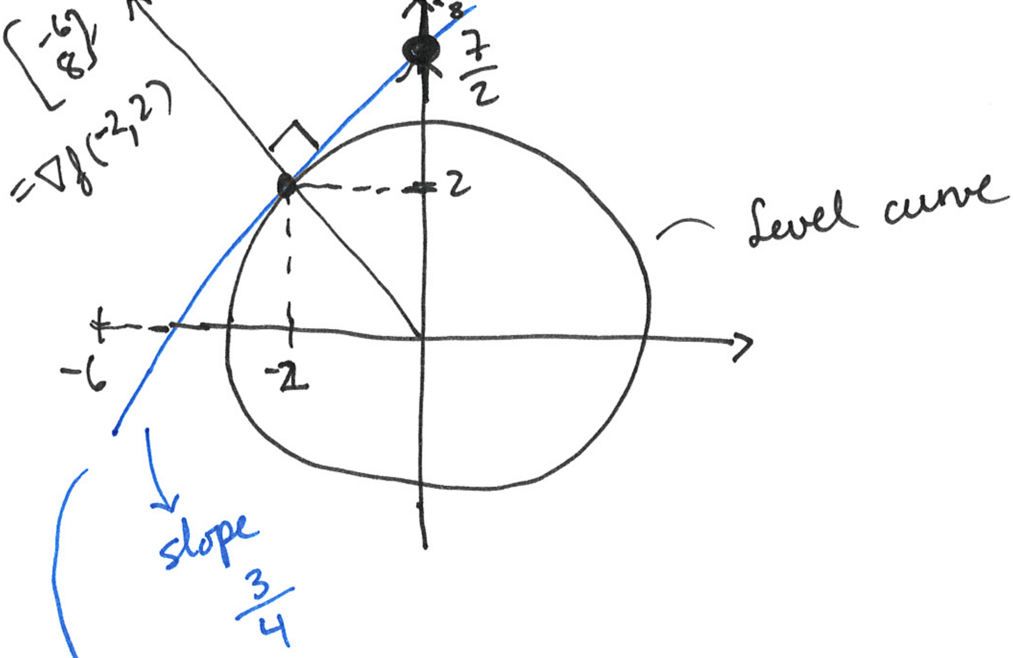


$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{3}{4}x + \frac{7}{2} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}_{\text{direction of vector}} x + \begin{bmatrix} 0 \\ \frac{7}{2} \end{bmatrix}$$

$$\nabla f(-2,2) \cdot \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = -6 + \cancel{8}^{2} \cdot \frac{3}{\cancel{4}}$$

$$= -6 + 6 = 0, \text{ so}$$

$\nabla f(-2,2) \perp \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}$ (90° angle), i.e. the tangent line of the level curve at $(-2,2)$.



Tangent line of level curve in $(-2,2)$

Directional derivatives

Def (Directional derivative): Let $f(x,y)$ be a function, $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ a 2-vector. Then,

$$f'_{\vec{a}} := \vec{a} \cdot \nabla f$$

Read: "The directional derivative of f wrt. $\vec{a}$"

Dot/inner product

Ex: $f(x,y)$ as before, $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Then,

$$f'_{\vec{a}} = \vec{a} \cdot \nabla f = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2x-2 \\ 2y+4 \end{bmatrix}$$

$$= 2(2x-2) + 1(2y+4)$$

$$= \ldots = \underline{4x+2y}$$

a number!

Can insert a point, say $(1,1)$

$$f'_{\vec{a}}(1,1) = 4\cdot 1 + 2\cdot 1 = \underline{\underline{6}}$$

More about the second derivative test and optimization

<u>NOTE</u>: If (x^*, y^*) is a stationary point with

$$H(f)(x^*, y^*) = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

f''_{xx} $\quad$ f''_{xy} $\quad$ f''_{yx} $\quad$ f''_{yy}

Then,

① If $\underbrace{AC - B^2}_{\text{det. of Hessian}} > 0$, then $AC > B^2 \geq 0$

since square

$\Rightarrow AC > 0$. But then,

$+\cdot+=+$

i) $A, C > 0$, so $\underbrace{A + C > 0}$

Local min. from 2nd deriv. test

OR

ii) $A, C < 0$, so $\underbrace{A + C < 0}$

$(-)\cdot(-)=(+)$

Local max. from 2nd deriv. test

det. > 0: Local max or local min.

Hence, the possible cases are:

i) $A, C > 0$; local min.

ii) $A, C < 0$; local max

$\underset{f''_{xx}}{A}, \underset{f''_{yy}}{C}$

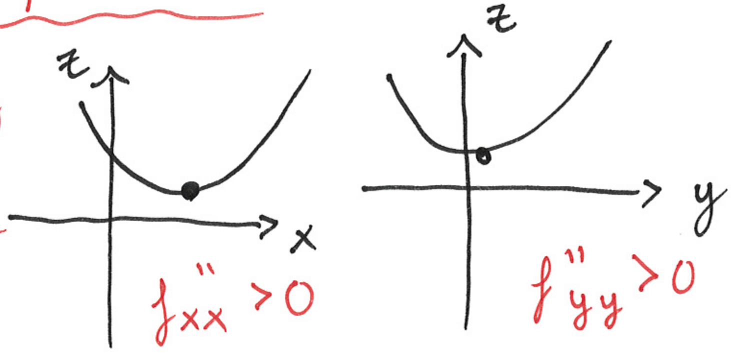
Recall: Second order derivative ≥ 0 ; Convex

—— " —— ≤ 0 ; Concave

Graphically: Cuts of the graph $z = f(x,y)$:

$A, C > 0$

Locally convex cuts:

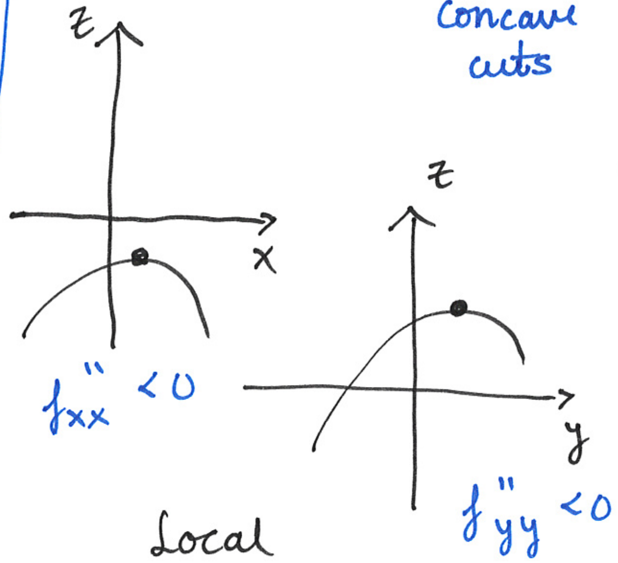


Local min.

(locally convex)

$A, C < 0$

Locally concave cuts



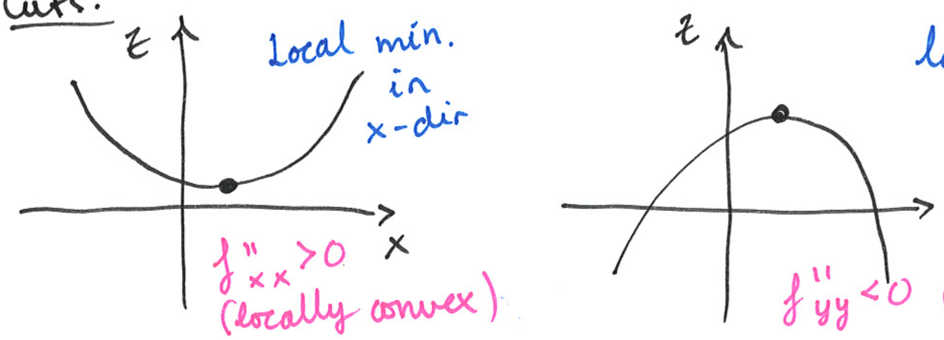
Local max.

(locally concave)

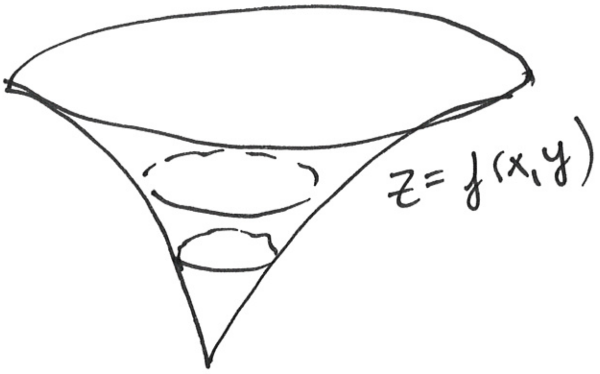
② If $\overbrace{AC - B^2}^{\det H} < 0$, a typical case is $\underbrace{A}_{f''_{xx}} > 0, \underbrace{C}_{f''_{yy}} < 0$

Def. of Hessian < 0: Saddle point

Cuts:



Ex: $f(x,y) = \sqrt{x^2+y^2}$, $D_f = \mathbb{R}^2 \Rightarrow$ No boundary points of D_f



Stationary points?

$$\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$$

From recap: $f'_x = \dfrac{x}{\sqrt{x^2+y^2}} = 0 \Rightarrow x = 0$

$$f'_y = \frac{y}{\sqrt{x^2+y^2}} = 0 \Rightarrow y = 0$$

Is $(0,0)$ a stationary pt? No! Because $\rightarrow$ division by 0

$f'_x(0,0)$ and $f'_y(0,0)$ are not defined.

$\Rightarrow$ There are no stationary points for f.

But $(0,0)$ is a critical point and candidate pt.

$f(0,0) = \sqrt{0^2+0^2} = 0$, which is the smallest value f can take. Hence, $(0,0)$ is a minimum pt. for f.

i.e., global min.