Course paper: March 13th - March 20th

EBA 1180
Spring 23

1) g) $\int_1^{e^2} \frac{\sqrt{\ln x}}{x}\,dx = \int \frac{\sqrt{u}}{\cancel{x}} \cdot \cancel{x}\,du$

SUBSTITUTION:

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$x = e^2 \Rightarrow u = \ln x = \ln e^2 = 2$

$x = 1 \Rightarrow u = \ln x = \ln 1 = 0$

$= \int_0^2 \sqrt{u}\,du = \int_0^2 u^{\frac{1}{2}}\,du$

$= \left[\frac{2}{3} u^{\frac{3}{2}}\right]_{u=0}^{2} = \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$

$= \frac{2}{3}\left(2\sqrt{2} - 0\right) = \underline{\underline{\frac{4}{3}\sqrt{2}}}$

$2^{\frac{3}{2}} = 2^{1+\frac{1}{2}} = 2^1 \cdot 2^{\frac{1}{2}} = 2\sqrt{2}$

General formula for a parabola

2.) $\underset{\sim}{P}$: $f(x) = a(x-2)^2 + 5$

since $x = 2$ is the axis of symmetry and $y = 5$ is the vertex

Parabola intersects the x-~~axis~~ axis in $\cancel{x} = 2 \pm \sqrt{5}$:

$f(\underbrace{2 \pm \sqrt{5}}_{x}) = 0$

$$a\left(\pm\sqrt{5}\right)^2 + 5 = 0$$

$$5a + 5 = 0$$

$$\underline{a = -1}$$

$$\Downarrow$$

P: $f(x) = 5 - (x-2)^2 = \underline{1 + 4x - x^2}$

General formula of hyperbola

H: $(x-0)(y-0) = c$ since $x=0$ and $y=0$ are asymptotes.

$$xy = c$$

$$y = \frac{c}{x}$$

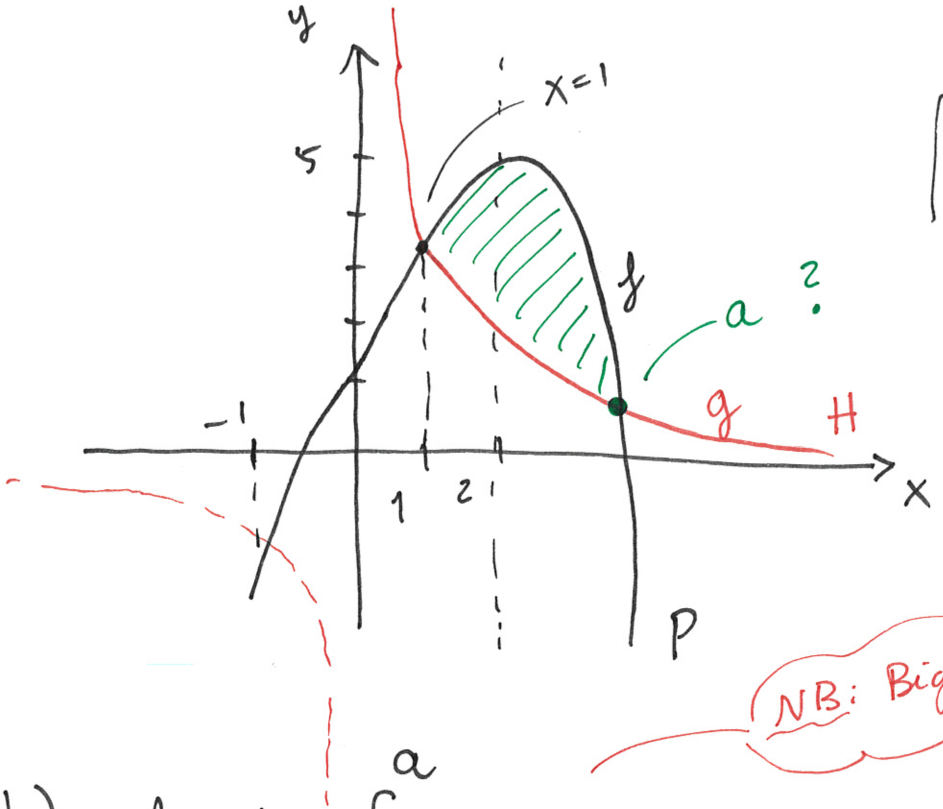
H: $g(x) = \frac{c}{x}$; What should c be?

Intersection in $x=1$: $f(1) = g(1)$

Hyperbola intersects parabola in $x=1$: $f=g$

$$1 + 4 - 1^2 = \frac{c}{1} \Rightarrow c = 4,$$

$$\underline{g(x) = \frac{4}{x}}$$



$\boxed{\text{Need to find a !}}$

NB: Biggest - smallest for area

b) $\text{Area} = \displaystyle\int_1^a f(x) - g(x)\,dx$

Find a: Intersection:

$$1 + 4x - x^2 = \frac{4}{x} \quad \Big| \cdot x$$

$$x + 4x^2 - x^3 = 4$$

$$x^3 - 4x^2 - x + 4 = 0$$

Polynomial division:

$\rightarrow (x^3 - 4x^2 - x + 4) : (x-1) = x^2 \,---$

$-(x^3 - x^2)$

$---$

$$(x-1)(x^2 - 3x - 4) = 0$$

$\underline{x=1}$ or $x^2 - 3x - 4 = 0$

$$(x-4)(x+1) = 0$$

$\underline{x=4}$, $\underline{x=-1}$ $\Rightarrow a = 4$

From figure

$$\text{Area} = \int_1^4 \underbrace{1 + 4x - x^2}_{f(x)} - \underbrace{\frac{4}{x}}_{g(x)} \, dx$$

$$= \left[x + 2x^2 - \frac{1}{3}x^3 - 4\ln|x| \right]_{x=1}^{4}$$

$$= \ldots \text{insert numbers} \ldots = \underline{\underline{12 - 8\ln 2}}$$

3) <u>Total cash flow:</u>

$$\int_0^{25} f(t)\,dt = \int_0^{25} 100 e^{\sqrt{t}}\,dt = \int 100 e^u \, 2\sqrt{t}\,du$$

<u>Substitution:</u>

$u = \sqrt{t}$

$du = \dfrac{1}{2\sqrt{t}}\,dt$

$t = 0 \Rightarrow u = \sqrt{0} = 0$

$t = 25 \Rightarrow u = \sqrt{25} = 5$

$$= \int_0^5 200 e^u u\,du = 200 \int_0^5 e^u u\,du$$

$$= 200 \left[ue^u - e^u \right]_{u=0}^{5}$$

Int. by parts:

$\int e^u u\,du = e^u u - \int e^u \cdot 1\,du$

$\int w' v = wv - \int w v'$

$$= 200\left(5e^5 - e^5\right) - 200\left(\underbrace{0e^{0}}_{0} - \overset{1}{e^0}\right)$$

$$= \ldots = \underline{\underline{800e^5 + 200}}$$

Expression for the net present value:

$$\int_0^{25} f(t)e^{-rt}\,dt = \int_0^{25} 100\,e^{rt}e^{-rt}\,dt$$

6) b) $x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 + c\vec{v}_4 = \vec{w}$

$\boxed{\text{TRICK}}$

$$\begin{array}{cccc|c} \multicolumn{5}{c}{} \\ \end{array}\left[\begin{array}{cccc|c} 5 & 3 & 1 & 3 & a \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right] \xrightarrow{R_1 - R_2} \sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 4 & 1 & 5 & 8 & b \\ 7 & 2 & 8 & 13 & c \end{array}\right]$$

$$\begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & b-4(a-b) \\ 0 & -12 & 36 & 48 & c-7(a-b) \end{array}\right] \quad -\frac{12}{7}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -4 & -5 & a-b \\ 0 & -7 & 21 & 28 & -4a+5b \\ 0 & 0 & 0 & 0 & \underbrace{c-7(a-b)-\frac{12}{7}(b-4(a-b))}_{(\star)} \end{array}\right]$$

$\vec{w}$ is a lin. comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ $\Leftrightarrow$ the lin. syst. is consistent $\Leftrightarrow$ $(\star) = 0$

$$c - 7(a-b) - \frac{12}{7}\left(b - 4(a-b)\right) = 0 \quad \Big| \cdot 7$$

$$7c - 49(a-b) - 12b + 48(a-b) = 0$$

$$7c - 12b - (a-b) = 0$$

$$-a - 11b + 7c = 0 \quad \Big| \cdot(-1)$$

$$a + 11b - 7c = 0$$

Conclusion: $\vec{w}$ is a lin. comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$

$$\iff \quad \underline{\underline{a + 11b - 7c = 0}}$$

7.) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\boxed{AX = XA}$

$$AX = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

For $AX = XA$:

$$\left.\begin{aligned} c &= b \\ d &= a \\ a &= d \\ b &= c \end{aligned}\right\} \Rightarrow \begin{aligned} &c, d \text{ free,} \\ &a = d \text{ and} \\ &b = c \end{aligned}$$

$$(a, b, c, d) = (d, c, c, d) = c(0, 1, 1, 0) + d(1, 0, 0, 1)$$

c, d are free

<u>Conclusion:</u>

$$X = c\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} + d\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I},$$

where c, d are free.

8.) a) $$\left.\begin{aligned} f'_x &= 2x - 4y - 4 = 0 \\ f'_y &= -4x + 10y + 4 = 0 \end{aligned}\right\} \quad \begin{aligned} 2x - 4y &= 4 \\ -4x + 10y &= -4 \end{aligned}$$

A linear system!

$$\begin{array}{cc} x & y \\ \downarrow & \downarrow \end{array}$$
$$\left[\begin{array}{cc|c} 2 & -4 & 4 \\ -4 & 10 & -4 \end{array}\right] \begin{matrix} \\ \leftarrow 2 \end{matrix} \sim \left[\begin{array}{cc|c} 2 & -4 & 4 \\ 0 & 2 & 4 \end{array}\right]$$

$\Rightarrow$ $2y = 4$, so $\underline{y = 2}$

$2x - 4y = 4$, so $2x = 4 + 4 \cdot 2 = 12$

$\underline{x = 6}$

<u>One stationary pt:</u> $(x, y) = (6, 2)$

$$H(f) = \begin{array}{c} \begin{array}{cc} f''_{xx} & f''_{xy} \end{array} \\ \begin{bmatrix} 2 & -4 \\ -4 & 10 \end{bmatrix} \\ \begin{array}{cc} f''_{yx} & f''_{yy} \end{array} \end{array}$$

$$\det H(f) = 2 \cdot 10 - (-4)(-4) = 20 - 16 = 4 > 0$$

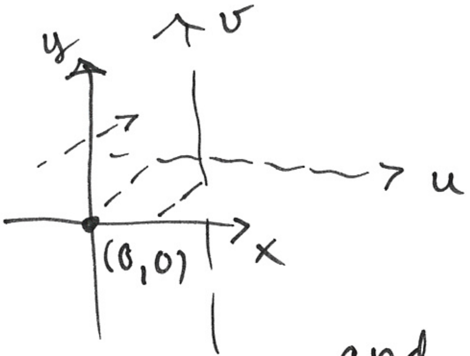
$$\operatorname{tr} H(f) = 2 + 10 = 12 > 0$$

Hence, <u>(6,2) is a local ~~max~~ minimum from the second derivative test.</u>

$$f(6,2) = 6^2 - 4\cdot 6\cdot 2 + 5\cdot 2^2 - 4\cdot 6 + 4\cdot 2 + 1$$
$$= \ldots = \underline{-7}$$

b) TRICK: Change of variables!

Make a change of variables $\begin{cases} u = x-6 \\ v = y-2 \end{cases}$ s.t.



the stationary pt. becomes $u = v = 0$. This gives $x = u+6$ and $y = v+2$ and:

$$\underline{f(u,v)} = (u+6)^2 - 4(u+6)(v+2) + 5(v+2)^2 - 4(u+6) + 4(v+2) + 1$$

$$= \ldots = \underbrace{u^2 - 4uv}_{\text{Complete the square}} + 5v^2 - 7$$

$$= \underbrace{(u-2v)^2}_{+} + \underbrace{v^2}_{+} - 7 \geq -7$$

Hence, $(u,v) = (0,0)$ or $(x,y) = (6,2)$ is a (global) minimum and $f_{min} = -7$.

The function f has no (local or global) max.