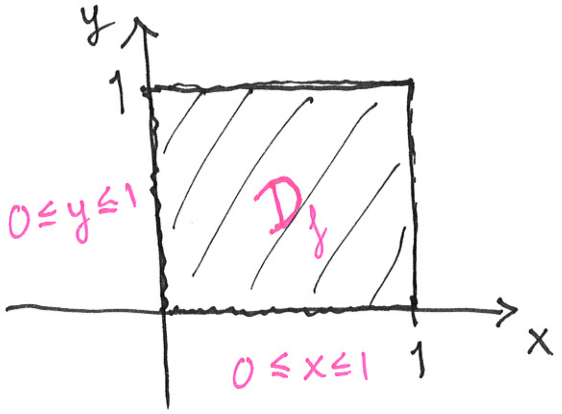
EBA1180
Spring 23

Warm up: $f(x,y) = x+y$, $\underbrace{0 \le x, y \le 1}_{D_f}$

What are the (global) max/min points?

Draw the domain D_f



See directly:

Maximum: $(1,1) \Rightarrow f(1,1) = 2$

Minimum: $(0,0) \Rightarrow f(0,0) = 0$

NOTE: f has both (global) min and max.

Q1: What if D_f: $0 < x, y < 1$

Q2: What if min/max $x+y$ over all of $\mathbb{R}^2$?

Constrained optimization and the extreme value theorem

- $f(x,y)$ is a continuous function on a set D in $\mathbb{R}^2$.

$\boxed{\text{EVT}}$

Extreme value theorem: If f is a continuous function on a compact set D in $\mathbb{R}^2$, then f has a maximum and a minimum on D.

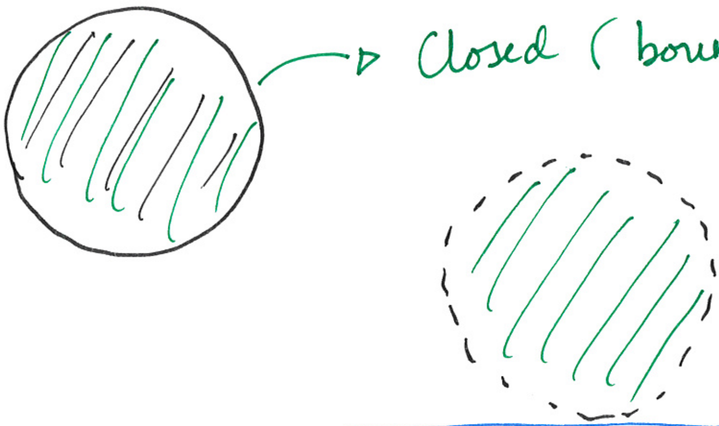
Def: (Compact set):
A subset D of $\mathbb{R}^2$ is compact if it is closed and bounded.

Def (Closed set): A subset D of $\mathbb{R}^2$ is closed if all boundary points of D are included in D.

NOTE: $=, \leq, \geq$; Closed

$<, >$; Not closed

Ex:

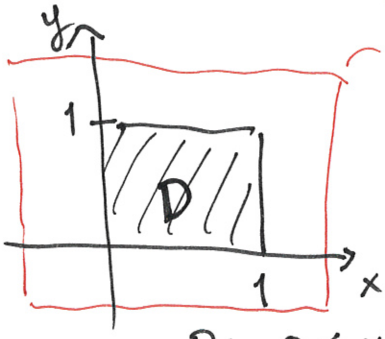


Closed (boundary included)

Not closed (boundary not included)

Def (Bounded set): A subset D of $\mathbb{R}^2$ is bounded if there exists a rectangle in $\mathbb{R}^2$ (with finite side lengths) that includes all of D.

Ex:



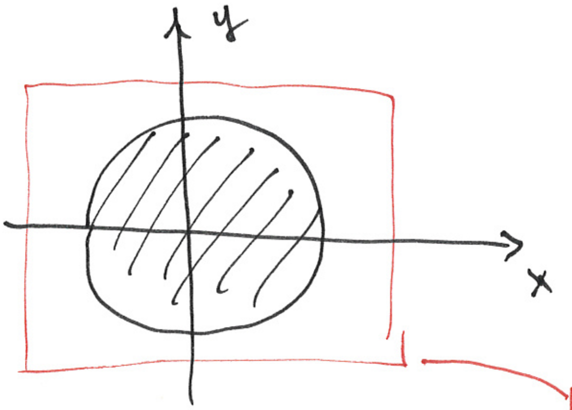
$D: \; 0 \leq x, y \leq 1$

Rectangle containing D $\Rightarrow$ Bounded ✓

Closed ✓ (contains boundary)

$\Downarrow$

Compact !



D: $x^2+y^2 \le 4$

Closed (contains boundary) ✓

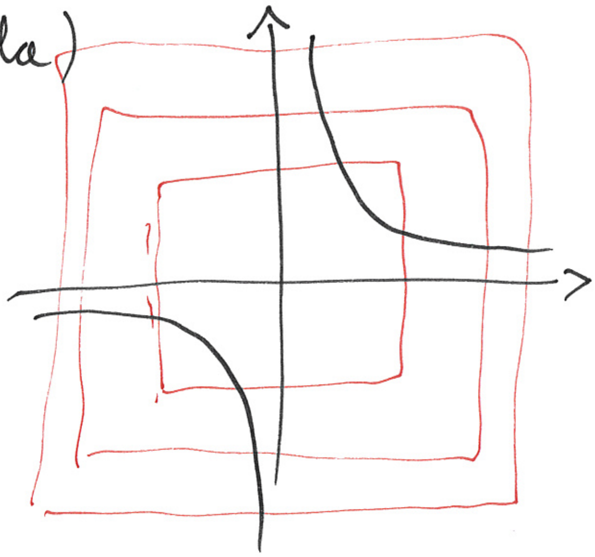
Bounded ✓

$\Downarrow$

Compact

D: $xy = 1$ (hyperbola)

$y = \frac{1}{x}$



Closed, but not bounded

Constrained optimization

max/min $\underbrace{f(x,y) = x^2+y^2}_{\text{Objective function}}$ when $\underbrace{0 \le x, y \le 1}_{\text{Constraints}}$

Constrained optimization

$D = \{(x,y) : 0 \le x, y \le 1\}$ in $\mathbb{R}^2$; Set of admissible points.

∞

max/min $f(x,y) = x^2 + y^2$

Unconstrained optimization

Unconstrained

max/ min $f(x,y)$

Candidate points

i) Stationary pts: $f'_x = 0$, $f'_y = 0$

ii) Other critical pts: f'_x or f'_y are not defined.

iii) Boundary points of D_f.

- (Local) classification: Second derivative test (local max, local min, saddle pt.)
- Must check: Are any of these points global max/ min?

Constrained

max/min $f(x,y)$ when (x,y) in D

Candidate points

i) Interior stationary points: $f'_x = 0$, $f'_y = 0$

ii) Other interior critical points: f'_x or f'_y not defined.

iii) Boundary points of D,

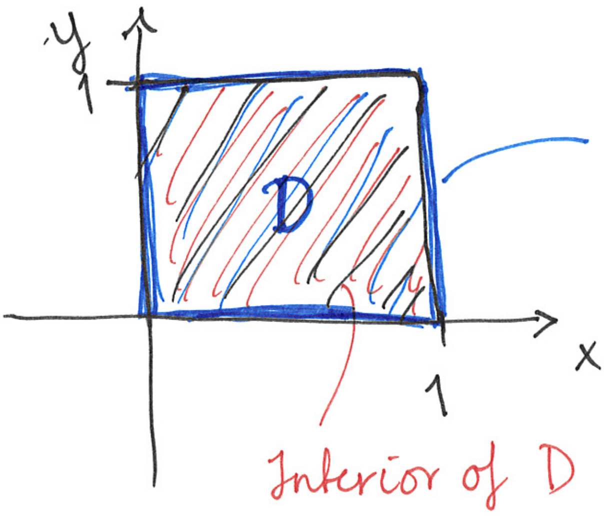
$D = \{(x,y): \text{all constraints are satisfied}\}$

EVT: If D is compact (closed and bounded), there is a global max/min.

- Determine whether the candidate pts are (global) max/min: Use EVT if D is compact.

Ex: $\max/\min\ f(x,y) = x^2 + y^2$ when

$$0 \le x, y \le 1$$



OBS: std. notation

∂D = boundary of D (the four sides of the square)

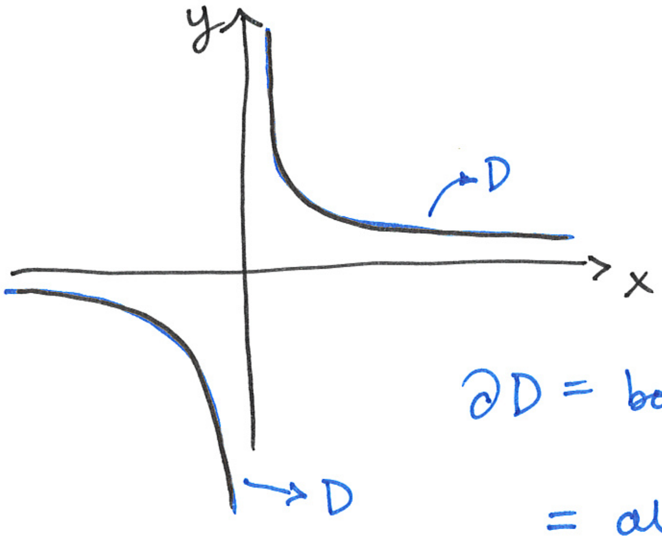
Max: Make x and y as large as possible $\Rightarrow$

$$x = y = 1, \quad f(1,1) = 2$$

Min: Make x and y as small as possible $\Rightarrow$

$$x = y = 0, \quad f(0,0) = 0$$

Ex: max/min $f(x,y) = x^2 + y^2$ when $xy = 1$



$xy = 1$

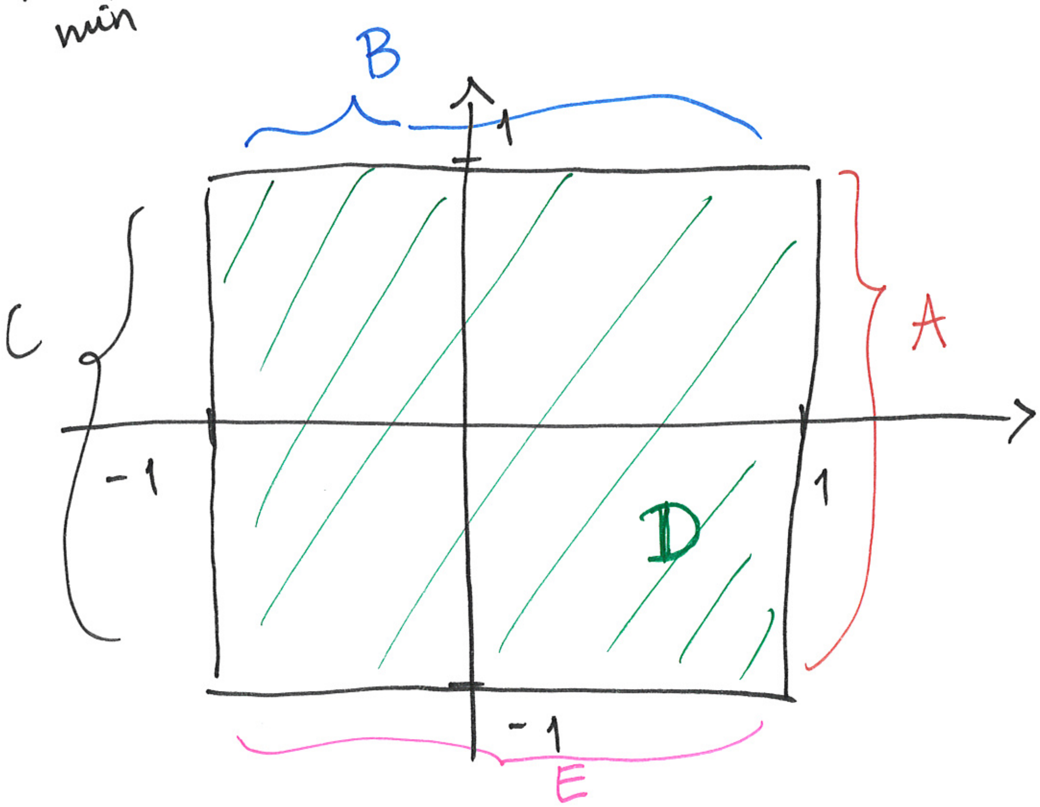
$y = \frac{1}{x}$

$x = 0$ impossible since $xy = 1$

∂D = boundary points of D

= all points on D

Ex: max/min $f(x,y) = x^2 + y^2$ when $-1 \le x, y \le 1$



Candidate points: i) ~~the~~ Interior stationary points:

$$f'_x = 2x = 0 \Rightarrow x = 0$$
$$f'_y = 2y = 0 \Rightarrow y = 0$$

$\}$ $(0,0)$ is an interior pt. of D

Candidate: $(x,y) = (0,0)$, $f(0,0) = 0$

ii) Other critical pts in the interior of D: None.

iii) Boundary points: ∂D = four sides

A: $x=1$, $-1 \le y \le 1$

B: $y=1$, $-1 \le x \le 1$

C: $x=-1$, $-1 \le y \le 1$

E: $y=-1$, $-1 \le x \le 1$

EVT: • f continuous? OK!

• D compact? Closed and bounded?
Yes! ✓ ✓

$\Rightarrow$ $\boxed{f(x,y) \text{ has a (global) max and min}}$

EVT holds

$\Downarrow$

There is a max. and a min. among the candidate points.

What is the max / min? Compare values of the candidate points:

i) Stationary: $(0,0) \Rightarrow f(0,0) = 0$

ii) Critical: None.

iii) Boundary:

A: $f(1,y) = 1 + y^2$, $-1 \leq y \leq 1$, See directly:

max: $f(1,1) = f(1,-1) = 2$, min: $f(1,0) = 1$

B: $f(x,1) = x^2 + 1$, $-1 \leq x \leq 1$. See dir:

max: $f(1,1) = f(-1,1) = 2$, min: $f(0,1) = 1$

C: $f(-1,y)$, $-1 \le y \le 1$

max: $f(-1,1) = f(-1,-1) = 2$

min: $f(-1,0) = 1$

E: $f(x,-1) = x^2+1$, $-1 \le x \le 1$

max: $f(1,-1) = f(-1,-1) = 2$

min: $f(0,-1) = 1$

Conclusion: D is compact, so there is a max/min. The highest value among the candidate points is

$$f_{max} = 2 \leftarrow \text{(global) max}$$

(at the max pts: $(1,1), (1,-1), (-1,1), (-1,-1)$)

The lowest value among the candidate points is

(global) min $\leftarrow$ $f_{min} = 0$ (at the min pt. $(0,0)$)

Alt. method side A: $x=1 \Rightarrow$

$f(1,y) = 1+y^2$, $-1 \le y \le 1$

$(1+y^2)' = 2y = 0 \Rightarrow y=0$

Candidates: $y=0$, $y=-1$, $y=1$