EBA 1180
Spring 23

Final exam 2022 (May 23rd)

Q1:

$$\begin{bmatrix} 2 & -6 & 4 & 6 \\ 3 & a & 7 & 2 \\ 1 & -2 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

a) Solve for $a = -12$:

$$\left[\begin{array}{cccc|c} 2 & -6 & 4 & 6 & 8 \\ 3 & -12 & 7 & 2 & 7 \\ 1 & -2 & 1 & 10 & 5 \end{array}\right] \begin{matrix} \\ \\ -1 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 3 & -12 & 7 & 2 & 7 \\ 1 & -2 & 1 & 10 & 5 \end{array}\right] \begin{matrix} -3 \\ \\ -1 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 0 & -2 & 14 & -2 \\ 0 & 2 & -2 & 14 & 2 \end{array}\right] \text{switch} \sim \left[\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 2 & -2 & 14 & 2 \\ 0 & 0 & -2 & 14 & -2 \end{array}\right]$$

Echelon form

Column with no pivot: w is free!

Equation form:

(1): $x - 4y + 3z - 4w = 3$

(2): $2y - 2z + 14w = 2$

(3): $-2z + 14w = -2$

$\Downarrow$

(3): $z = 7w + 1$

$\downarrow$

(2): $2y = 2 + 2(7w+1) - 14w = 4 \Rightarrow y = 2$

$\downarrow$

$(y=2) \downarrow \quad \downarrow (z = 7w+1)$

(3): $x = 3 + 4 \cdot 2 - 3 \cdot (7w+1) + 4w$

$= \cancel{3} + 8 - 21w - \cancel{3} + 4w$

$= 8 - 17w$

Solution: $(x, y, z, w) = (8-17w, 2, 1+7w, w)$ with w free.

b) Determine a (if any) s.t. system has no solutions:

$$\left[\begin{array}{cccc|c} 2 & -6 & 4 & 6 & 8 \\ 3 & a & 7 & 2 & 7 \\ 1 & -2 & 1 & 10 & 5 \end{array}\right] \quad -1$$

$$\sim \left[\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 3 & a & 7 & 2 & 7 \\ 1 & -2 & 1 & 10 & 5 \end{array}\right] \quad -3 \quad -1$$

$$\sim \left[\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & a+12 & -2 & 14 & -2 \\ 0 & 2 & -2 & 14 & 2 \end{array}\right] \quad \text{switch}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 2 & -2 & 14 & 2 \\ 0 & a+12 & -2 & 14 & -2 \end{array}\right] \cdot \frac{1}{2}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 1 & -1 & 7 & 1 \\ 0 & a+12 & -2 & 14 & -2 \end{array}\right] \quad \cdot\, -(a+12)$$

$$\sim \left[\begin{array}{cccc|c} 1 & -4 & 3 & -4 & 3 \\ 0 & 1 & -1 & 7 & 1 \\ 0 & 0 & a+10 & \underbrace{14-7(a+12)}_{=-7(a+10)\ :\ (\star)} & \underbrace{-2-(a+12)}_{=-14-a\ :\ (\star\star)} \end{array}\right]$$

some calculations

The system has <u>no solutions</u> if the last row of the matrix is $[0 \ \ 0 \ \ 0 \ \ 0 \mid \text{non-zero}]$.

Hence, we need:

$$\left.\begin{array}{r} a+10=0 \\ -7(a+10)=0 \end{array}\right\} \iff \underline{a=-10}$$

$$(\sim) \quad -14-a \neq 0$$

<u>Does $(\sim)$ hold for $a=-10$?</u>

$-14-a = -14+10 = -4 \neq 0$, so $(\sim)$ holds.

<u>Conclusion:</u> For $a=-10$, the system has <u>no solutions</u>.

Q2:

a) $\int_0^1 6\sqrt{x} - 11x\sqrt[5]{x}\,dx$

$= \int_0^1 6x^{\frac{1}{2}} - 11x^1 x^{\frac{1}{5}}\,dx$

$= \int_0^1 6x^{\frac{1}{2}} - 11x^{\frac{6}{5}}\,dx$

$= \left[6\frac{2}{3}x^{\frac{3}{2}} - \cancel{11}\frac{5}{\cancel{11}}x^{\frac{11}{5}}\right]_{x=0}^{1}$

$= \left[4x^{\frac{3}{2}} - 5x^{\frac{11}{5}}\right]_{x=0}^{1}$

$= 4\cdot\underbrace{1^{\frac{3}{2}}}_{1} - 5\cdot\underbrace{1^{\frac{11}{5}}}_{1} - (\underbrace{4\cdot0^{\frac{3}{2}}}_{0} - \underbrace{5\cdot0^{\frac{11}{5}}}_{0})$

$= 4 - 5 = \underline{\underline{-1}}$

b) $\int \frac{21-x}{9-x^2}\,dx = \int \frac{3}{3-x} + \frac{4}{3+x}\,dx$

$= 3\ln|3-x|\cdot\frac{1}{(-1)} + 4\ln|3+x|\cdot\frac{1}{1} + C$

$= 4\ln|3+x| - 3\ln|3-x| + C$

Partial fractions:

$\frac{21-x}{9-x^2} = \frac{A}{3-x} + \frac{B}{3+x} \quad \Big| \cdot (3-x)(3+x)$

$21 - x = A(3+x) + B(3-x)$

$21 - x = (3A+3B) + x(A-B)$

$\Downarrow$

$3A + 3B = 21 \quad | :3 \Rightarrow A + B = 7$

$A - B = -1$

$+\ A - B = -1$

$2A = 6$

$\Rightarrow A = 3$

$B = 7 - A = 7 - 3 = 4$

c) $\int \frac{1}{1-\sqrt{x}}\,dx = \int \frac{1}{u}(-2\sqrt{x})\,du$

Substitution:

$u = 1-\sqrt{x}$

$du = \frac{-1}{2\sqrt{x}}\,dx$

$\sqrt{x} = 1-u$

$dx = -2\sqrt{x}\,du$

$= \int \frac{-2(1-u)}{u}\,du$

$= \int \frac{-2+2u}{u}\,du$

$= \int 2 - \frac{2}{u}\,du$

$= 2u - 2\ln|u| + C$

SHOW EXERCISE!

$= 2(1-\sqrt{x}) - 2\ln|1-\sqrt{x}| + C$

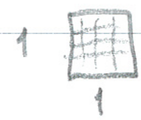
(Graph shows $f'(x)$. Estimate $f(1)-f(0)$:)

d) $f(1)-f(0) = \int_0^1 f'(x)\,dx = -A$ where A is the area between the graph of $f'(x)$ and the x-axis on $[0,1]$

since f is an antiderivative of f' by def.

$A \approx 4$ squares $= 4 \cdot \underbrace{\frac{1}{16}} = \frac{1}{4} = 0{,}25$

each sq. has area $\frac{1}{16}$ since there are 16 sq in the



-box

Conclusion: $f(1) - f(0) = \underline{\underline{-A}} \approx -0{,}25$.

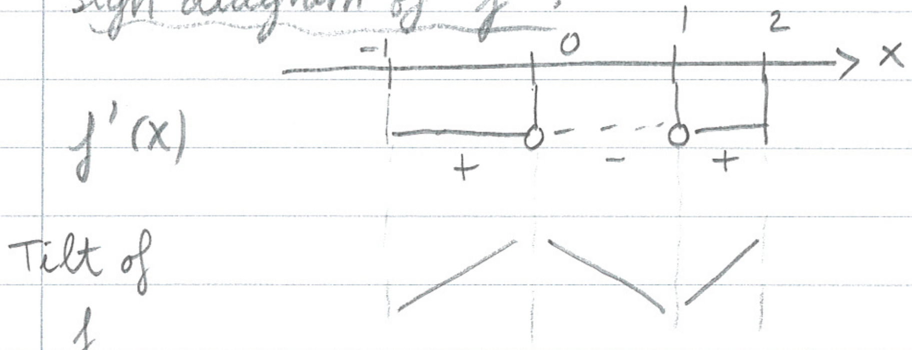
e) (Find x-coordinates of max/min pts. of f, if exist:)

max/min $f(x)$

Candidates: i) Boundary points: $\underline{x=-1}$, $\underline{x=2}$

ii) Stationary points: $f'(x) = 0 \Rightarrow \underline{x=0}$, $\underline{x=1}$

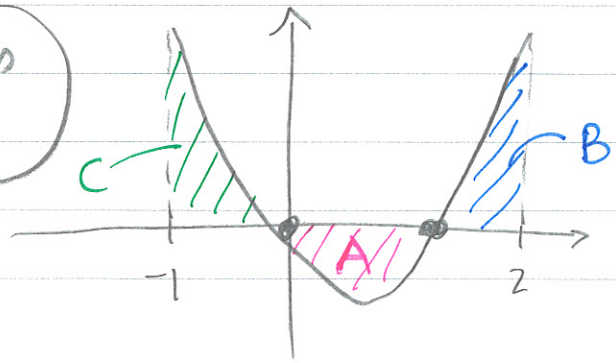
Sign diagram of f':



Possible max (from sign diagram): $\underline{x=0}$, $\underline{x=2}$

Which is bigger? $f(2) - f(0) = \int_0^2 f'(x)\,dx = -A + B$

Same method as in d)



But, from figure, $B > A$. Hence

$$f(2) - f(0) = B - A > 0$$

$B > A$

$B - A > 0$

$$\Updownarrow$$

$$\boxed{f(2) > f(0)}$$

Possible min (from sign diagram): $x = -1$, $x = 1$:

$$f(1) - f(-1) = \int_{-1}^{1} f'(x)\,dx = C - A$$

Sign diagram

Figure

But, from the figure, $C > A$. Hence,

$$f(1) - f(-1) = C - A > 0$$

$C > A$

$C - A > 0$

$$\Updownarrow$$

$$\boxed{f(1) > f(-1)}$$

Conclusion: $x = 2$ is the max. point for f.
$x = -1$ is the min. point for f.

Q3:

$$A = \begin{bmatrix} a & 1 & 2 \\ 1 & a & 1 \\ 2 & 1 & a \end{bmatrix}$$

(See we're asked to find $|A|$ for general a in b) : ~~Do~~ this to begin with to solve a) quicker)

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & 1 & 2 \\ 1 & a & 1 \\ 2 & 1 & a \end{vmatrix} = a(a^2-1) - 1(a-2) + 2(1-2a)$$

$$= a^3 - a - a + 2 + 2 - 4a$$

$$= \underline{a^3 - 6a + 4}$$

$(0^3 - 6 \cdot 0 + 4)$

a) $a=0$: $|A| = 4 \neq 0$, so A has an inverse when $a=0$.

FORMULA! Not on sheet.

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$C_{11} = -1$, $C_{12} = 2$, $C_{13} = 1$

$C_{21} = 2$, $C_{22} = -4$, $C_{23} = 2$

$C_{31} = 1$, $C_{32} = 2$, $C_{33} = -1$

(Alt: A symmetric $\Rightarrow$ adj(A) symmetric for lower triangle

$$= \frac{1}{4} \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}^T$$

$$= \frac{1}{4} \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

b) Have shown:

$$|A| = a^3 - 6a + 4;$$

$$|A| = 0 \Leftrightarrow a^3 - 6a + 4 = 0$$

3rd order eq: Must guess for solution:

$a=0$: Doesn't work ($4 \neq 0$)

$a=1$: $1^3 - 6\cdot 1 + 4 = 1 - 6 + 4 = -1$

$a=2$: $2^3 - 6\cdot 2 + 4 = 8 - 12 + 4 = \underline{0}$

$\Rightarrow$ $a = 2$ is a solution! So $(a-2)$ is a factor in $|A|$.

Polynomial division: $(a^3 - 6a + 4) : (a-2) = a^2 + 2a - 2$

$$\begin{array}{r} -(a^3 - 2a^2) \\ \hline 2a^2 - 6a + 4 \\ -(2a^2 - 4a) \\ \hline -2a + 4 \\ -(-2a + 4) \\ \hline 0 \end{array}$$

$$\Rightarrow |A| = (a-2)\underbrace{(a^2 + 2a - 2)}_{\downarrow} = 0$$

Solve $a^2 + 2a - 2 = 0$: $a = \dfrac{-2 \pm \sqrt{4 - 4\cdot 1\cdot(-2)}}{2}$

$$= \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm 2\sqrt{3}}{2} = \underline{-1 \pm \sqrt{3}}$$

$$|A| = (a-2)(a-(-1+\sqrt{3}))(a-(-1-\sqrt{3}))$$

Conclusion: so $|A| = 0$ for $a = 2$, $a = -1-\sqrt{3}$ and

$$a = -1+\sqrt{3}.$$

c) (Solve $A\vec{x} = \vec{0}$:) ⊳ Find $\vec{x}$ & a s.t. this has a solution

For $a \neq 2, -1 \pm \sqrt{3}$, $|A| \neq 0$. Hence,

$$A\vec{x} = \vec{0} \Leftrightarrow A^{-1}A\vec{x} = A^{-1}\vec{0}$$

$$I\vec{x} = \vec{0}$$

$$\vec{x} = \vec{0}$$

So in this case, $\vec{x} = 0$ is the only solution.

We are left with $a = 2$, $a = -1 \pm \sqrt{3}$.
Simplest to calculate with $a = 2$:

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Gaussian elimination:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]\begin{array}{c} -2 \\ \\ -2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right]\begin{array}{c} \\ -1 \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad \begin{aligned} x + 2y + z &= 0 \\ -3y &= 0 \\ (z \text{ free}) & \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\underline{y = 0} \\ &\downarrow \\ &x = 0 - 2y - z \\ &\underline{x = -z} \end{aligned}$$

Conclusion: $(x, y, z) = (-z, 0, z)$ with z free.

For example, with $z = 1$, we get:

$$(x, y, z) = (-1, 0, 1) \text{ and}$$

$$A \cdot \vec{x} = \vec{0} \text{ when } (x, y, z) = (-1, 0, 1) \text{ and } a = 2.$$

Infinitely many other solutions (z can be anything!).

Q4: $f(x,y) = x^2y - 5xy^2 + xy^3$

a) Stationary pts:

(1): $f'_x = 2xy - 5y^2 + y^3 = y(2x - 5y + y^2) = 0$

(2): $f'_y = x^2 - 10xy + 3xy^2 = x(x - 10y + 3y^2) = 0$

$\Rightarrow$ FROM eq(1): $y = 0$ OR $2x - 5y + y^2 = 0$

FROM eq.(2): $x = 0$ OR $x - 10y + 3y^2 = 0$

4 combinations:

1: $y = 0, x = 0$; $(x,y) = (0,0)$.

2: $y = 0, x - 10y + 3y^2 = 0$: $x - 10\cdot 0 + 3\cdot 0^2 = 0$

$x = 0$

$\Rightarrow (x,y) = (0,0)$

3: $2x - 5y + y^2 = 0, x = 0$; $-5y + y^2 = 0$

$y(-5 + y) = 0$

$y = 0$ OR $y = 5$

$\Rightarrow (x,y) = (0,0)$ OR $(x,y) = (0,5)$

4: $2x - 5y + y^2 = 0,\ x - 10y + 3y^2 = 0$:

$$\left.\begin{array}{l} 2x - 5y + y^2 = 0 \\ x - 10y + 3y^2 = 0 \end{array}\right\} \to x = 10y - 3y^2$$

$$2(10y - 3y^2) - 5y + y^2 = 0$$

$$20y - 6y^2 - 5y + y^2 = 0$$

$$-5y^2 + 15y = 0$$

$$5y(-y + 3) = 0$$

$\underline{y = 0}$ OR $\underline{y = 3}$

$x = 10 \cdot 0 - 3 \cdot 0^2$

$\underline{x = 0}$

$x = 10 \cdot 3 - 3 \cdot 3^2$

$= 30 - 27$

$\underline{= 3}$

So:

$\underline{(x, y) = (0, 0)}$ OR $\underline{(x, y) = (3, 3)}$

Conclusion: The stationary points are

$(0, 0),\ (0, 5),\ (3, 3)$

Not (0,0)

b) (Classify the stationary points:)

$$H(f) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix} = \begin{bmatrix} 2y & 2x-10y+3y^2 \\ 2x-10y+3y^2 & -10x+6xy \end{bmatrix}$$

Hessian: Not on sheet!

(0,5):

$$H(f)(0,5) = \begin{bmatrix} 10 & 25 \\ 25 & 0 \end{bmatrix}$$

$$\det(H(f)(0,5)) = 10 \cdot 0 - 25^2 = -625 < 0$$

$\Rightarrow$ (0,5) is a saddle point.

Second derivative test (on sheet)

(3,3):

$$H(f)(3,3) = \begin{bmatrix} 6 & 3 \\ 3 & 24 \end{bmatrix}$$

$$\det(H(f)(3,3)) = 6 \cdot 24 - 3 \cdot 3 = 135 > 0$$

$$\mathrm{tr}(H(f)(3,3)) = 6 + 24 = 30 > 0$$

$\Rightarrow$ (3,3) is a local min.

Second derivative test

Q5: max $f(x,y) = x + 3y$ when $x^2 - 6x + 9y^2 + 18y + 9 = 0$

a) (Sketch D; bounded?)

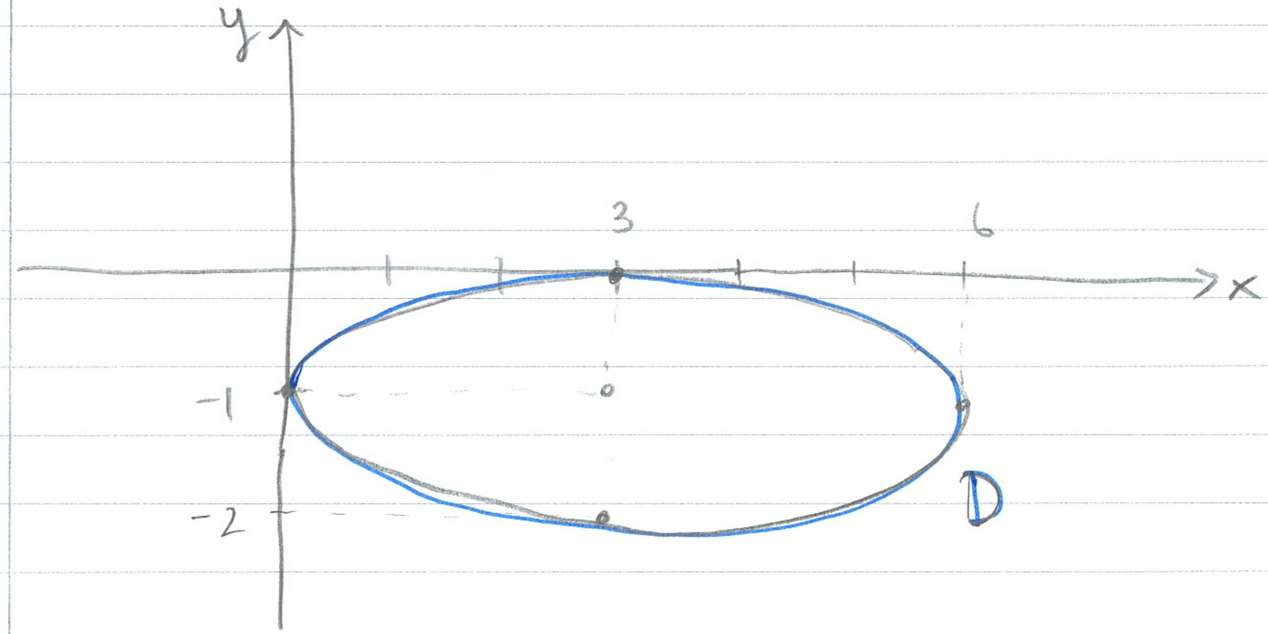
D: $x^2 - 6x + 9y^2 + 18y + 9 = 0$

Try to complete the squares!

$$x^2 - 6x + 9 + 9(y^2 + 2y + 1) = 9 \quad \Big| :9$$

$$\frac{(x-3)^2}{9} + \frac{(y+1)^2}{1} = 1 \Rightarrow$$

Ellipse, center $(3,-1)$, half-axes $a = \sqrt{9} = 3$ and $b = \sqrt{1} = 1$.



D is bounded since $0 \le x \le 6$ and $-2 \le y \le 0$.

(NOT on sheet)

b) $\mathcal{L} = x + 3y - \lambda(x^2 - 6x + 9y^2 + 18y + 9)$

$$\boxed{\begin{aligned} \mathcal{L}'_x &= 1 - \lambda(2x - 6) = 0 \quad (1) \\ \mathcal{L}'_y &= 3 - \lambda(18y + 18) = 0 \quad (2) \\ C:\ & x^2 - 6x + 9y^2 + 18y + 9 = 0 \end{aligned}}$$

(on sheet)

(NOT on sheet)

By the extreme value theorem, there is a max: f is continuous, D is closed ($=$) and bounded (by a)).

Since D is an ellipse, there is a unique tangent at every point $\rightarrow$ No admissible points with a degenerate constraint. Alt: Check this by analyzing $g'_x = 0$, $g'_y = 0$ and C: holding.

(on sheet)

$\Downarrow$

$\boxed{\text{The maximum is the ordinary candidate point with the greatest value}}$

From 1) and 2):

$$\lambda = \frac{1}{2x-6} = \frac{3}{18y+18}$$

(*):

$$18y + 18 = 3(2x-6)$$

$$18(y+1) = 6(x-3)$$

$$\Rightarrow x - 3 = 3(y+1)$$

→ Note: this is only OK for $x \neq 3$ and $y \neq 1$.

However, these are impossible because in this case, 1) & 2) are not satisfied (the product would be 0 → so we can safely divide like this

Recall C:

$$\frac{(x-3)^2}{9} + \frac{(y+1)^2}{1} = 1$$

in ellipsis's std. form.

FROM a)

INSERT

$$\frac{3^2(y+1)^2}{\cancel{9}} + \frac{(y+1)^2}{1} = 1$$

$$2(y+1)^2 = 1$$

$$(y+1)^2 = \frac{1}{2}$$

$$\underline{y = -1 \pm \sqrt{\frac{1}{2}}}$$

$$\Downarrow$$

INSERT y BACK

$$x - 3 = 3(\cancel{-1} \pm \sqrt{\frac{1}{2}})$$

$$\underline{x = \cancel{3} \pm 3\sqrt{\frac{1}{2}}}$$

$$\lambda = \frac{1}{2(x-3)} = \frac{1}{2\cdot 3(\pm\sqrt{\frac{1}{2}})}$$

From (*)

$$= \pm \frac{1}{6\sqrt{\frac{1}{2}}}$$

Candidate points:

$$(x, y; \lambda) = \left(\underline{3 + 3\sqrt{\tfrac{1}{2}}, -1 + \sqrt{\tfrac{1}{2}}}; \frac{1}{6\sqrt{\frac{1}{2}}}\right),$$

$f = 6\sqrt{\frac{1}{2}}$

$$\left(\underline{3 - 3\sqrt{\tfrac{1}{2}}, -1 - \sqrt{\tfrac{1}{2}}}; -\frac{1}{6\sqrt{\frac{1}{2}}}\right)$$

$f = -6\sqrt{\frac{1}{2}}$

Conclusion:

$$\underline{f_{max} = 6\sqrt{\tfrac{1}{2}}} = \frac{3\cdot 2}{\sqrt{2}} = \frac{3\sqrt{2}\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = \underline{3\sqrt{2}}$$

at $\left(3 + 3\sqrt{\frac{1}{2}}, -1 + \sqrt{\frac{1}{2}}; \frac{1}{6\sqrt{\frac{1}{2}}}\right)$.

Q6.) max $f(x, y) = y$ when $x(x^2 + y^2) = x^2 - y^2$

a) (Admissible pts. with degenerate constraint?)

$$g(x, y) = x^3 + xy^2 - x^2 + y^2 = 0$$

Degenerate constraint

(1): $g'_x = 3x^2 + y^2 - 2x = 0$

(2): $g'_y = 2xy + 2y = 0 \Rightarrow 2y(x+1) = 0 \Rightarrow$

$y = 0$ OR $x = -1$

$y = 0$: (1) $\Rightarrow 3x^2 + 0^2 - 2x = 0$

$x(3x-2) = 0$

$x = 0$ or $x = \frac{2}{3}$

$\Rightarrow (0,0)$ or $(0, \frac{2}{3})$

$x = -1$: (1) $\Rightarrow 3 + y^2 + 2 = 0$

$y^2 = -5$; Impossible!

$\Rightarrow$ No points.

Are the points admissible?

$(0,0)$: $g(0,0) = 3 \cdot 0^2 + 0^2 - 2 \cdot 0 = 0$; Yes.

$(0, \frac{2}{3})$: $g(0, \frac{2}{3}) = 3 \cdot 0 + \left(\frac{2}{3}\right)^2 - 2 \cdot 0$

$= \frac{4}{9} \neq 0$; No.

Conclusion: There is one admissible point with degenerate constraint:

$(x,y) = (0,0)$.

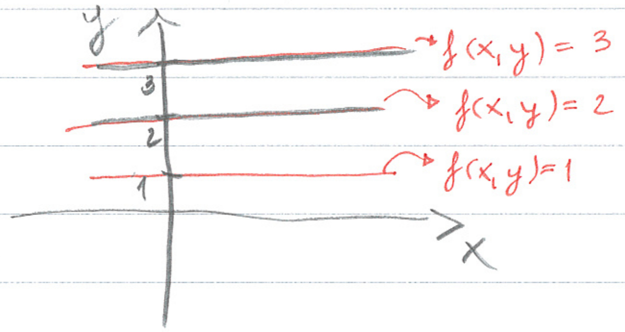
b) (Why are the pts. on the curve
$C = \{(x,y) : x(x^2+y^2) = x^2 - y^2\}$
with horisontal tangent exactly the pts.
that satisfy the Lagrange cond.?)

Note: The Lagrange problem is

max $f(x,y) = y$ when
$x(x^2+y^2) = x^2 - y^2$

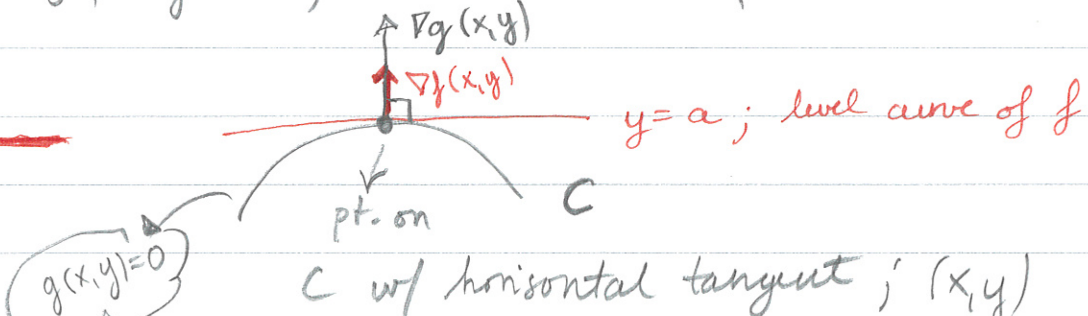
What are the level curves of f?

$f(x,y) = a$
$y = a$; horisontal, straight lines



NB:
($f(x,y)$ is a tilted plane floating above the xy-axis)

A point on C: $x(x^2+y^2) = x^2 - y^2$ has horizontal tangent $\Longleftrightarrow$ The level curve of f, $y = a$, meets this point at a tangent:



$$\Updownarrow$$

The point, (x, y), is on C (so constraint holds) and satisfies

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{(see figure prev. pg.)}$$

$$\Updownarrow$$

$$\nabla f(x,y) - \lambda \nabla g(x,y) = 0$$

$$\Updownarrow$$

$$\nabla \mathcal{L}(x,y) = 0$$

From def. of Lagrangian

$$\Updownarrow$$

$$\begin{bmatrix} \mathcal{L}'_x(x,y) \\ \mathcal{L}'_y(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Updownarrow$$

$$\left.\begin{array}{ll} & \mathcal{L}'_x(x,y) = 0 \\ \text{and} & \mathcal{L}'_y(x,y) = 0 \end{array}\right\} \text{FOC}$$

But then the constraint and first order conditions of Lagrangian holds, which happens $\Leftrightarrow$

(x, y) is an ordinary candidate point of the Lagrange problem.

(READ FROM START TO FINISH & THE RESULT IS PROVED.)