

... if I couldn't formulate a problem in economic theory mathematically, I didn't know what I was doing.

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Lecture 11 – 12

Sec. 6.3.1-3, 5.4-5, 4.7

Increasing/decreasing functions. Circles, ellipses. Polynomial functions.

Here are recommended exercises from the textbook [SHSC].

- Section 6.3 exercise 3
- Section 5.4 exercise 1, 3
- Section 5.5 exercise 1-6
- Section 4.7 exercise 4

Problems for the exercise session
Wednesday 5 Oct. from 12 in B2-065

Problem 1 Determine the equations of the circles in figure 1.

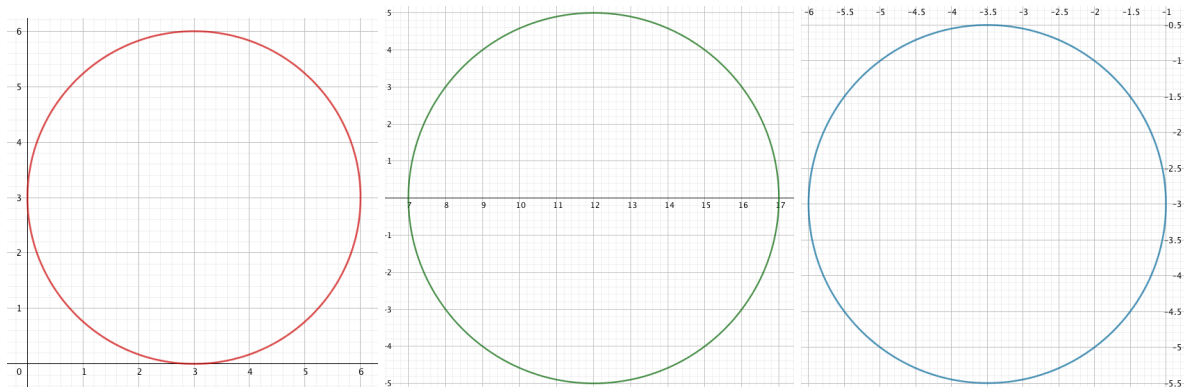


Figure 1: Circles a-c

Problem 2 Determine the center S and the radius r of the circles.

- a) $(x - 3)^2 + (y - 4)^2 = 5$
- b) $(x + 1)^2 + y^2 = 3$
- c) $(3x - 2)^2 + (3y - 4)^2 = 9$
- d) $x^2 + y^2 - 4x - 10y = -25$
- e) $x^2 + y^2 + 6x - 12y = -44$
- f) $25x^2 + 25y^2 - 20x - 30y = -12$

Problem 3 Determine the equations of the ellipses in figure 2.

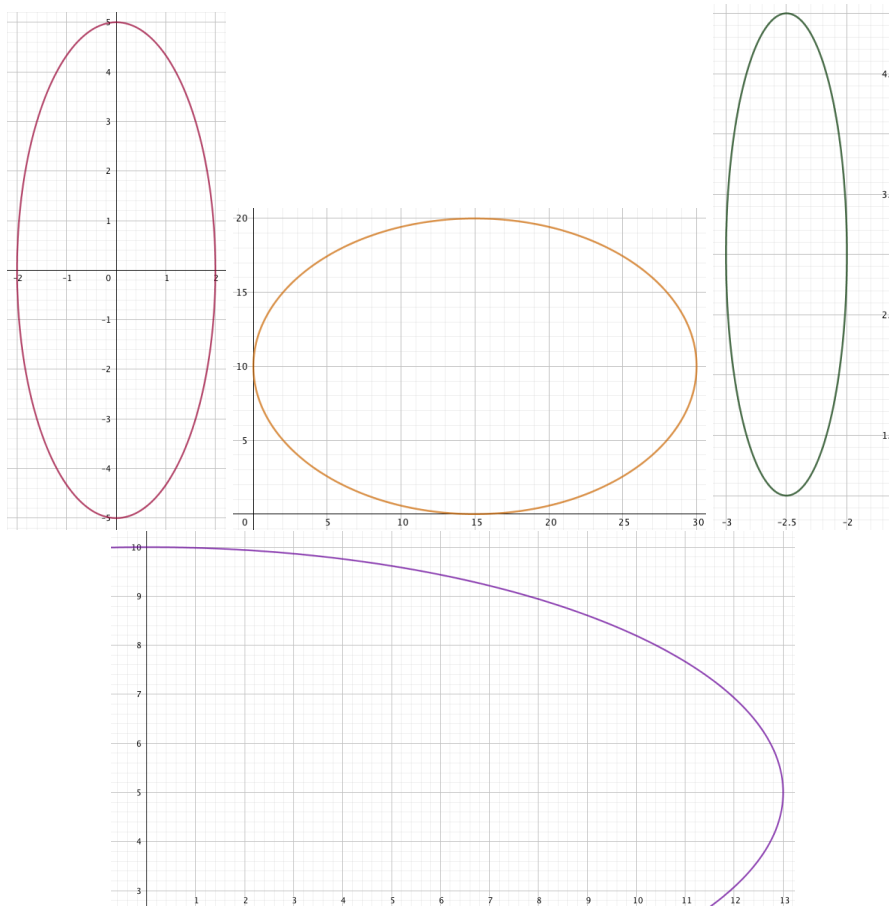


Figure 2: Ellipses a-d

Problem 4 Determine the center S and the semi-axes of the ellipse. Draw a sketch of the ellipse.

a) $\frac{x^2}{9} + \frac{y^2}{16} = 1$

b) $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{16} = 1$

c) $16(x-1)^2 + 9(y-2)^2 = 144$

d) $\frac{x^2}{2} + y^2 - 6y = -8$

e) $9x^2 + 18x + 4y^2 = 27$

f) $4x^2 + 9y^2 - 16x + 18y = 11$

g) $25x^2 + 4y^2 - 100x - 40y = -100$

Problem 5 Give elementary arguments for the statements.

a) $f(x) = x^2$ with $x \geq 0$ is strictly increasing.

b) $f(x) = \sqrt{x}$ is strictly increasing.

c) $f(x) = \frac{1}{x}$ with $x > 0$ is strictly decreasing.

Problem 6 Determine the intersection points of

a) the line $3x + 2y = 12$ and the line $-3x + 2y = -6$

b) the line $2x + y = 6$ and the ellipse in Problem 4a

Problem 7 Determine which expressions (below) and graphs (in figure 3) that belong together.

1) $x^4 - 8x^3 + 24x^2 - 32x + \frac{161}{10}$

2) $\frac{x^5}{10} - \frac{3x^4}{2} + \frac{17x^3}{2} - \frac{45x^2}{2} + \frac{137x}{5} - 10$

3) $-x^3 + 6x^2 - 11x + 7$

4) $x^4 - 10x^3 + 35x^2 - 50x + 26$

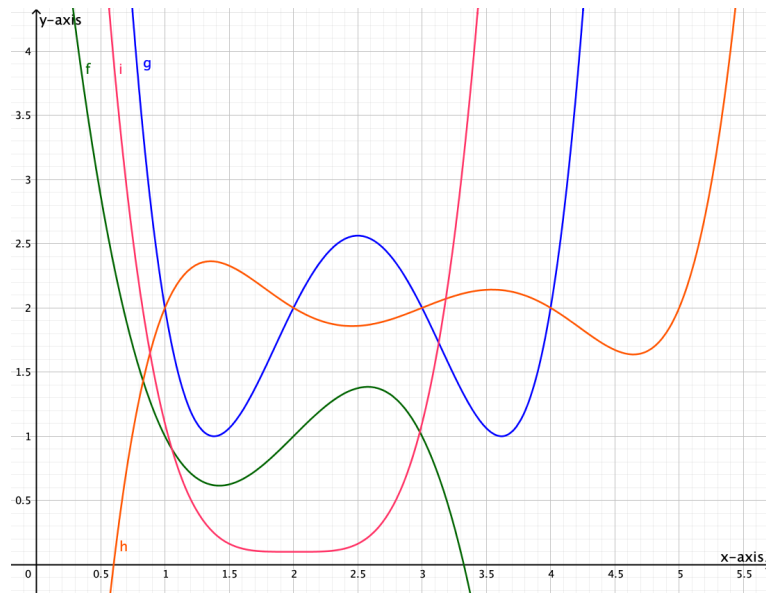


Figure 3: The graphs of four polynomial functions

Answers

Problem 1

a) $(x-3)^2 + (y-3)^2 = 9$

b) $(x-12)^2 + y^2 = 25$

c) $(x+3.5)^2 + (y+3)^2 = 6,25$

Problem 2

a) $S = (3, 4), r = \sqrt{5}$

b) $S = (-1, 0), r = \sqrt{3}$

c) $S = (\frac{2}{3}, \frac{4}{3}), r = 1$

d) $S = (2, 5), r = 2$

e) $S = (-3, 6), r = 1$

f) $S = (\frac{2}{5}, \frac{3}{5}), r = \frac{1}{5}$

Problem 3

a) $\frac{x^2}{4} + \frac{y^2}{25} = 1$

b) $\frac{(x-15)^2}{225} + \frac{(y-10)^2}{100} = 1$

c) $4(x+2.5)^2 + \frac{(y-3)^2}{4} = 1$

d) $\frac{x^2}{169} + \frac{(y-5)^2}{25} = 1$

Problem 4

a) $S = (0, 0)$, semi-axes $a = 3, b = 4$

b) $S = (1, 2)$, semi-axes $a = 3, b = 4$

c) $S = (1, 2)$, semi-axes $a = 3, b = 4$

d) $S = (0, 3)$, semi-axes $a = \sqrt{2}, b = 1$

e) $S = (-1, 0)$, semi-axes $a = 2, b = 3$

f) $S = (2, -1)$, semi-axes $a = 3, b = 2$

g) $S = (2, 5)$, semi-axes $a = 2, b = 5$

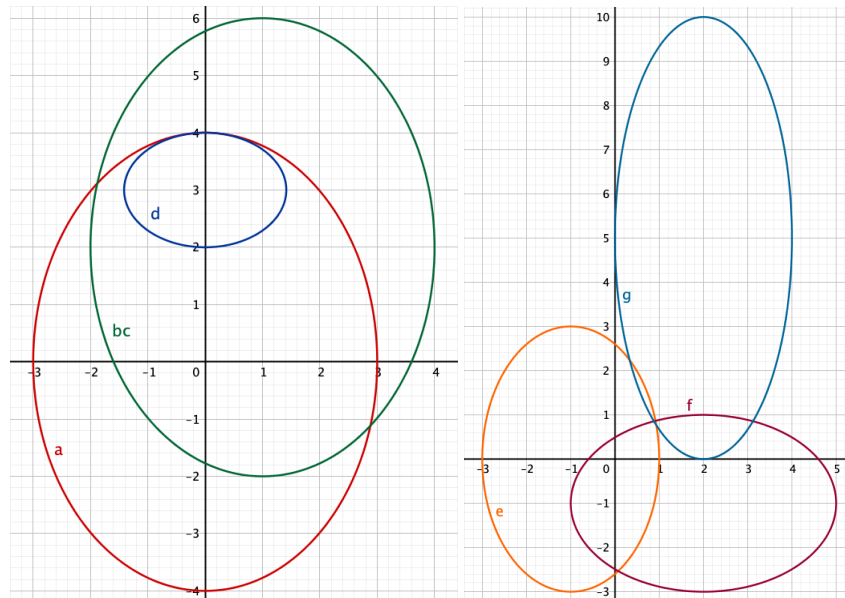


Figure 4: Ellipses a-d and e-g

Problem 5

- a) Suppose $0 \leq x_1 < x_2$. Then $x_2 = x_1 + k$ for a positive constant k . Then $f(x_2) = (x_1 + k)^2 = x_1^2 + 2kx_1 + k^2$. The product and the sum of two positive numbers are positive numbers, hence $2kx_1 + k^2$ is a positive number. Then $f(x_1) = x_1^2 < x_1^2 + 2kx_1 + k^2 = f(x_2)$ and $f(x) = x^2$ for $x \geq 0$ is strictly increasing.
- b) We divide each side of the inequality $x_1 < x_2$ with the positive number x_2 and get the inequality $\frac{x_1}{x_2} < 1$. The square root of a number which is less than 1 is itself less than 1, i.e. $\sqrt{\frac{x_1}{x_2}} < 1$. But $\sqrt{\frac{x_1}{x_2}} = \frac{\sqrt{x_1}}{\sqrt{x_2}}$. We get the inequality $\frac{\sqrt{x_1}}{\sqrt{x_2}} < 1$ and when we multiply each side with the positive number $\sqrt{x_2}$ we get the inequality $f(x_1) = \sqrt{x_1} < \sqrt{x_2} = f(x_2)$. Hence $f(x) = \sqrt{x}$ is strictly increasing.
- c) We divide each side of the inequality $x_1 < x_2$ with the positive number x_2 and get the equivalent inequality $\frac{x_1}{x_2} < 1$. Then we divide this inequality by the positive number x_1 and get $f(x_2) = \frac{1}{x_2} < \frac{1}{x_1} = f(x_1)$. Hence $f(x) = \frac{1}{x}$ for $x > 0$ is strictly decreasing.

Problem 6

a) $(3, \frac{3}{2})$

b) $(3, 0)$ and $(\frac{15}{13}, \frac{48}{13})$

Problem 7

$$f(x) = -x^3 + 6x^2 - 11x + 7$$

$$g(x) = x^4 - 10x^3 + 35x^2 - 50x + 26$$

$$h(x) = \frac{x^5}{10} - \frac{3x^4}{2} + \frac{17x^3}{2} - \frac{45x^2}{2} + \frac{137x}{5} - 10$$

$$i(x) = x^4 - 8x^3 + 24x^2 - 32x + \frac{161}{10}$$