EBA1180 Mathematics for Data Science autumn 2022

Exercises

I came to the position that mathematical analysis is not one of the many ways of doing economic theory: it is the only way.

R. Lucas

Lecture 19 – 20

Sec. 7.1, 6.9, 8.6-7:

Implicit differentiation. The second order derivative, convex/concave functions.

Here are recommended exercises from the textbook [SHSC].

Section **7.1** exercise 1, 4, 6, 7a
Section **6.9** exercise 1-4
Section **9.6** exercise 1-4, 6a
Section **8.6** exercise 1-4

Problems for the exercise session
Wednesday 9 Nov. from 12 – 17 in B2-065

Problem 1 Find an expression for y' in terms of y and x by implicit differentiation. Find all solutions for y with $x = a$ and determine the expression for the tangent function in each of these points.

a) $x^2 + 25y^2 - 50y = 0$ and $a = 4$

b) $x^{3.27}y^{1.09} = 1$ and $a = 1$

c) $x^4 - x^2 + y^4 = 0$ and $a = \frac{\sqrt{2}}{2}$

d) $x^3 - 3xy + y^2 = 0$ and $a = 2$

Problem 2 in figure 1 you see the graphs of the implicit defined curves in Problem 1. Determine the curves and the equations which belong together. Also draw the tangents in Problem 1.

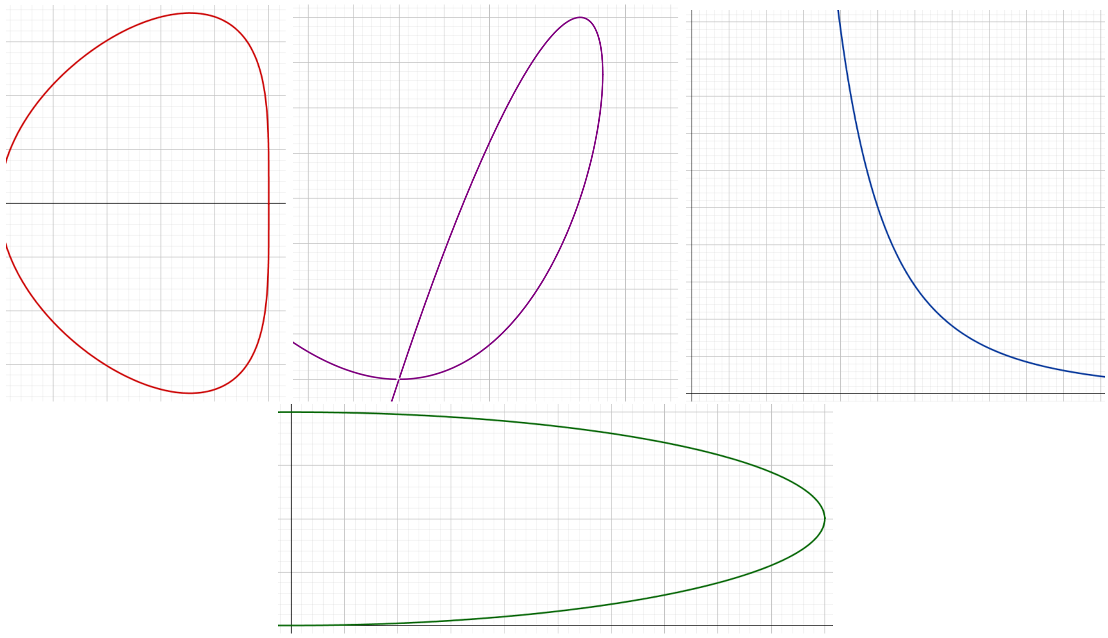

Figure 1: Four implicitly defined curves

Problem 3 Make a sketch of the graphs of **TWO** different functions $f(x)$ with the given data. One of the functions should be *strictly increasing*. Note: You are not supposed two find any algebraic expression!

a) $f''(x)$ is negative for $x < 5$ and positive for $x > 5$

b) $f''(x)$ is positive for $x < 10$, negative for $10 < x < 15$ and positive for $x > 15$

Problem 4 in figure 2 you see the graph of $f''(x)$. Determine if the statement is true or false.

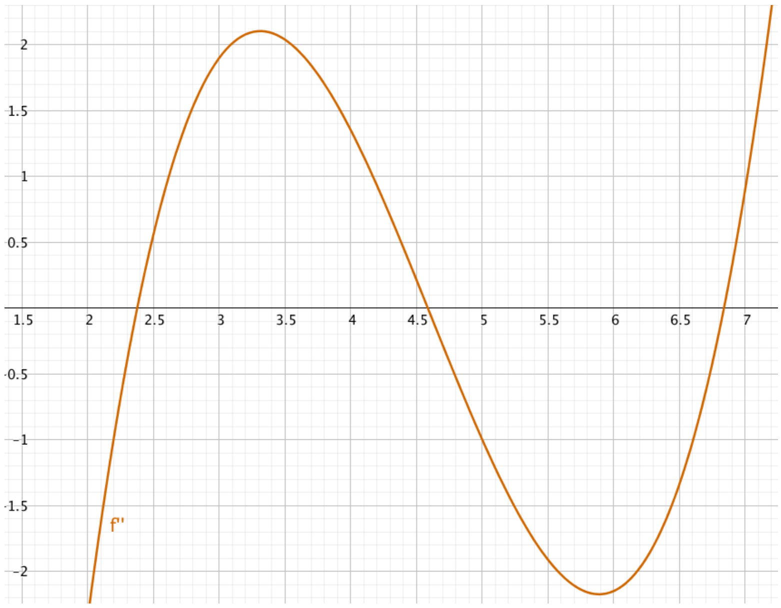

Figure 2: The graph of $f''(x)$

a) $f''(2.5) > f''(4)$

b) $f(x)$ is convex for $3 \leqslant x \leqslant 4$

c) $f(x)$ has no inflection points between 5.5 and 6

d) $f(x)$ has two inflection points for $2 \leqslant x \leqslant 7$

e) $f(x)$ is concave for $6 \leqslant x \leqslant 6.5$

f) $f'(4)$ is the maximum of $f'(x)$ for $x \in [3, 4]$

g) $f'(x)$ decreases in the interval $[4, 5]$

h) $f'(x)$ increases faster around $x = 2.5$ than around $x = 3$

i) $f(4)$ has to be positive

j) $f'(2.5) < f'(4.5)$

k) $f(x)$ must have at least one minimum point

Problem 5 In figure 3 you see the graphs of $f(x)$, $f'(x)$ and $f''(x)$ in the same coordinate system. Determine which is the graph of $f(x)$, of $f'(x)$ and of $f''(x)$ in (a-c).

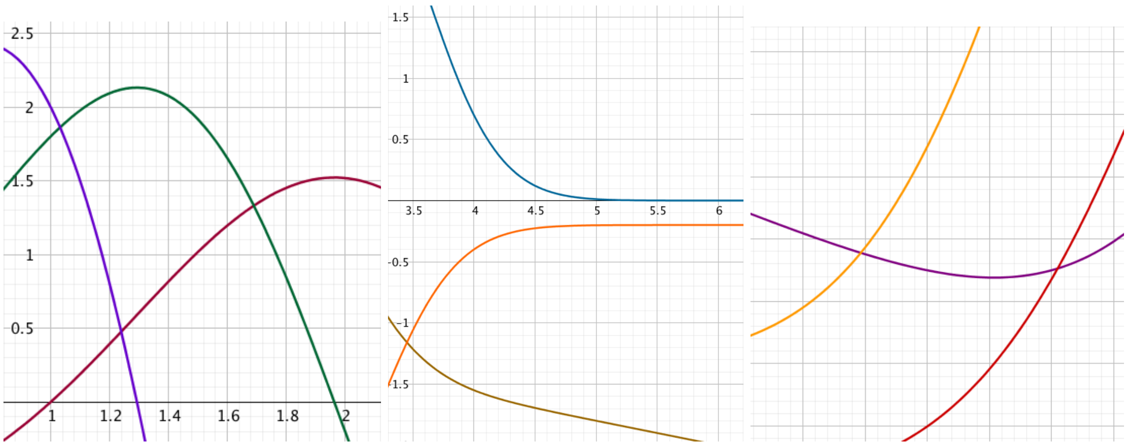

Figure 3: (a-c): The graphs of $f(x)$, $f'(x)$ and $f''(x)$

Problem 6 Calculate $f'(x)$ and $f''(x)$, solve the equation $f''(x) = 0$, determine where $f(x)$ is convex and concave, and determine the inflection points (if any).

a) $f(x) = x^4 - 8x^3 + 18x^2 + 1$ b) $f(x) = \ln(x^2 - 2x + 2) - \frac{x}{4} + 1$

c) $f(x) = e^{\frac{-x^2}{2}} + x + 1$ d) $f(x) = x^5 - 10x^4 + 30x^3 + 2$

Problem 7 Determine the expressions for the tangent functions at the inflection points in Problem 6.

Problem 8 Determine (local) minimum and maximum points for the function $f(x)$. Explain why these points give (global) minimum/maximum for $f(x)$ by using convexity/concavity of the function. Calculate the minimum/maximimum of the function.

a) $f(x) = \ln(-x^2 + 14x - 45)$ with $D_f = \langle 5, 9\rangle$ b) $f(x) = \frac{-1}{x(x-6)}$ with $D_f = \langle 0, 6\rangle$ c) $f(x) = e^{x(x-4)}$ with $D_f = \mathbb{R}$ (all real numbers)

Problem 9 (Multiple choice spring 2018, problem 11)
We consider the function $f(x) = 4\sqrt{x}\ln(x)$. Which statement is true?
(A) The function f has one inflection point
(B) The function f has several inflection points
(C) The function f is concave
(D) The function f is convex
(E) I choose not to solve this problem.

Problem 10 Compute the expression for the derivative of $f(x)$.

a) $f(x) = \sqrt{x^2 - 7x + 13}$ b) $f(x) = xe^{0.1x^2}$

c) $f(x) = (2x + 5)^{100}$ d) $f(x) = \dfrac{\ln(x)}{x}$

Answers

Problem 1
a) $y' = \frac{-x}{25(y-1)}$, for $x = 4$: $y = \frac{2}{5}$ or $y = \frac{8}{5}$ which gives the tangent functions $h_1(x) = \frac{4}{15}x - \frac{2}{3}$ and $h_2(x) = -\frac{4}{15}x + \frac{8}{3}$
b) $y' = \frac{-3y}{x}$, for $x = 1$: $y = 1$ which gives the tangent function $h(x) = -3x + 4$
c) $y' = \frac{x(1-2x^2)}{2y^3}$, for $x = \frac{\sqrt{2}}{2}$: $y = \pm\frac{\sqrt{2}}{2}$ which gives the tangent functions $h_1(x) = \frac{\sqrt{2}}{2}$ and $h_2(x) = -\frac{\sqrt{2}}{2}$
d) $y' = \frac{3(y-x^2)}{2y-3x}$, for $x = 2$: $y = 4$ or $y = 2$ which gives the tangent functions $h_1(x) = 4$ and $h_2(x) = 3x - 4$

Problem 2

a) Green b) Blue c) Red d) Purple

Problem 3
Compare with other students, ask the learning assistants!

Problem 4

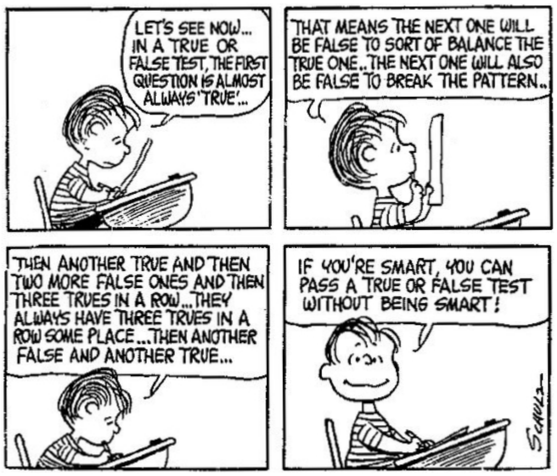

Figure 4: True or false, or opposite

Problem 5

a) $f(x)$: Dark red, $f'(x)$: Green

b) $f(x)$: Olive, $f'(x)$: Orange

c) $f(x)$: Violet, $f'(x)$: Red

Problem 6

a) $f'(x) = 4x^3 - 24x^2 + 36x$ and $f''(x) = 12(x-1)(x-3)$. $f''(x) = 0$ has solutions $x = 1$ and $x = 3$. $f(x)$ is convex in the interval $\langle\infty\,,1]$, $f(x)$ is concave in the interval $[1\,,3]$, and $f(x)$ is convex in the interval $[3\,,\infty\rangle$. Hence $x = 1$ and $x = 3$ are inflection points.

b) $f'(x) = \frac{2x-2}{(x-1)^2+1} - \frac{1}{4}$ and $f''(x) = \frac{-2x(x-2)}{[(x-1)^2+1]^2}$. $f''(x) = 0$ has solutions $x = 0$ and $x = 2$. $f(x)$ is concave in the interval $\langle\infty\,,0]$, $f(x)$ is convex in the interval $[0, 2]$, and $f(x)$ is concave in the interval $[2\,,\infty\rangle$. Hence $x = 0$ and $x = 2$ are inflection points.

c) $f'(x) = -xe^{\frac{-x^2}{2}} + 1$ and $f''(x) = (x+1)(x-1)e^{-\frac{x^2}{2}}$, $f''(x) = 0$ has solutions $x = \pm 1$, $f(x)$ is convex in the interval $\langle\infty\,,-1]$, $f(x)$ is concave in the interval $[-1, 1]$, and $f(x)$ is convex in the interval $[1\,,\infty\rangle$. Hence $x = -1$ and $x = 1$ are inflection points.

d) $f'(x) = 5x^4 - 40x^3 + 90x^2$ and $f''(x) = 20x(x-3)^2$. $f''(x) = 0$ has solutions $x = 0$ and $x = 3$ (a double root). $f(x)$ is concave in the interval $\langle\infty\,,0]$ and $f(x)$ is convex in the interval $[0\,,\infty\rangle$. Hence $x = 0$ is the only inflection point.

Problem 7

a) Inflection point tangents: $h_1(x) = 16x - 4$ and $h_3(x) = 28$

b) Inflection point tangents: $h_0(x) = -1.25x + \ln(2) + 1$ and $h_2(x) = 0.75x + \ln(2) - 1$

c) Inflection point tangents: $h_{-1}(x) = (1 + e^{-0.5})x + 2e^{-0.5} + 1$ and $h_1(x) = (1 - e^{-0.5})x + 2e^{-0.5} + 1$

d) Inflection point tangent: $h_0(x) = 2$

Problem 8

a) $f'(x) = \frac{2(7-x)}{-x^2+14x-45}$ which changes sign from + to − at $x = 7$. $f''(x) = \frac{-2[(x-7)^2+4]}{(-x^2+14x-45)^2}$ is negative for all x, so $f(x)$ is concave, max: $f(7) = 2\ln(2) = 1.39$

b) $f'(x) = \frac{2x-6}{x^2(x-6)^2}$ which changes sign from − to + at $x = 3$. $f''(x) = \frac{-6[(x-3)^2+3]}{x^3(x-6)^3}$ is positive for all $x \in \langle 0\,,6\rangle$, so $f(x)$ is convex, min: $f(3) = \frac{1}{9} = 0.11$

c) $f'(x) = 2(x-2)e^{x(x-4)}$ which changes sign from − to + at $x = 2$.
$f''(x) = 4[(x-2)^2 + \frac{1}{2}]e^{x(x-4)}$ is positive for all x, so $f(x)$ is convex, min: $f(2) = e^{-4} = 0.02$

Problem 9 A

Problem 10

a) $f'(x) = \dfrac{2x-7}{2\sqrt{x^2-7x+13}}$

b) $f'(x) = \dfrac{1}{5}(x^2+5)e^{0.1x^2}$

c) $f'(x) = 200(2x+5)^{99}$

d) $f'(x) = \dfrac{1-\ln(x)}{x^2}$