Exercise session problems

Problem 1.
Find A^{-1}, if possible:

a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 7 & -1 \\ 4 & 2 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$

d) $A = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ f) $A = \begin{pmatrix} 7 & 1 & 4 \\ -2 & 1 & -2 \\ 3 & 3 & 0 \end{pmatrix}$

Problem 2.
Determine the values of a such that the inverse matrix of A exists, and compute A^{-1} in these cases:

a) $A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & 1 & a \\ 0 & a & 1 \\ 0 & 0 & 2 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & 1 & a \\ 1 & 3 & 1 \\ a & 1 & 1 \end{pmatrix}$

Problem 3.
Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} t & 0 & 1 \\ 0 & t & 0 \\ 1 & 0 & t \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}$$

a) Solve the system for $t = 2$.

b) Determine how many solutions the system has for different values of t.

c) Find the inverse matrix A^{-1} when it exists, and use this to solve the system in these cases.

Problem 4.
Write the expressions as simple as possible:

a) $(A+B)^2$ b) $(A^TA)^T$ c) $A(3B-C)+(A-2B)C+2B(C+2A)$

d) $A^{-1}(BA)$ e) $(BAB^{-1})^2 \cdot B^2$ f) $(A-B)(C-A)+(C-B)(A-C)+(C-A)^2$

Problem 5.
Assume that A and B is 3×3-matrices with $|A| = 2$ and $|B| = -5$. Compute:

a) $\det(AB)$ b) $\det(3A)$ c) $\det(-2B^T)$ d) $\det(2A^{-1}B)$

Problem 6.
We consider the linear system $A \cdot \mathbf{x} = \mathbf{b}$ with parameter a, given by

$$A = \begin{pmatrix} 1 & a & 4 \\ 2a & 8 & 12 \\ 5 & 10 & 16 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 40 \\ 51 \end{pmatrix}$$

a. Use Gaussian elimination to solve the linear system when $a = 2$. Mark the pivot positions.

b. Compute $\det(A)$, and determine all values of a such that $\det(A) = 0$.

c. Find A^{-1} when $a = 3$.

d. Show that $A^7 \cdot \mathbf{x} = \mathbf{b}$ has exactly one solution for $a = -1$, and express the solution $\mathbf{x}$ via A and $\mathbf{b}$.

Problem 7.
Let A be a $n \times n$ matrix. An elementary row operation $A \to B$ corresponds to multiplication with an $n \times n$-matrix E from the left, such that $B = E \cdot A$. Then, E is called the elementary matrix of the row operation $A \to B$. Find the elementary matrices of the following row operations on 3×3-matrices:

a) Switch the two final rows

b) Multiply the second row by -1

c) Add 2 times row one to row two

d) Add -2 times row three to row one

Explain why all elementary matrices are invertible, and why a quadratic matrix is invertible if and only if it is a product of elementary matrices.

Problem 8.
Use elementary row operations to find the inverse matrix of A, if it exists. Check your answer by comparing with the determintant and the adjoint matrix of A.

a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 3 & 4 & 1 \end{pmatrix}$

Problem 9.
We consider the linear system $A \cdot \mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} a & 1 & a \\ 1 & 2 & 3 \\ a & 3 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -a \\ 3 - a \end{pmatrix}$$

and a is a parameter.

a) **(6p)** Solve the linear system when $a = 1$.

b) **(6p)** Find the determinant $\det(A)$, and determine the values of a such that $\det(A) = 0$.

c) **(6p)** Determine all values of a such that $A \cdot \mathbf{x} = \mathbf{b}$ has infinitely many solutions.

d) **(6p)** Compute $A^2 - 3A$ when $a = 1$.

Optional: Exercises from the Norwegian textbook

Textbook [E]:	Eriksen, *Matematikk for økonomi og finans*
Exercise book [O]:	Eriksen, *Matematikk for økonomi og finans - Oppgaver og Løsningsforslag*

Exercises:	[E] 6.6.1 - 6.6.6
Solution manual:	Se [O] Kap 6.6

Answers to exercise session problems

Problem 1.

a) $A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ b) $A^{-1} = \frac{1}{18}\begin{pmatrix} 2 & 1 \\ -4 & 7 \end{pmatrix}$ c) A^{-1} not defined

d) $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{pmatrix}$ e) $A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$ f) A^{-1} not defined

Problem 2.

a) $A^{-1} = \dfrac{1}{1-a^2}\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}$ for $a \neq -1, 1$ b) $A^{-1} = \dfrac{1}{6a}\begin{pmatrix} 2a & -2 & 1-a^2 \\ 0 & 6 & -3 \\ 0 & 0 & 3a \end{pmatrix}$ for $a \neq 0$

c) $A^{-1} = \dfrac{1}{(1-a)(1+3a)}\begin{pmatrix} 2 & a-1 & 1-3a \\ a-1 & 1-a^2 & a-1 \\ 1-3a & a-1 & 2 \end{pmatrix}$ for $a \neq -1/3, 1$

Problem 3.

a) $(x,y,z) = (2/3,0,2/3)$

b) Infinitely many solutions for $t = 0$ and $t = 1$, no solutions for $t = -1$, and one solution for $t \neq -1,0,1$

c) $A^{-1} = \dfrac{1}{t(t^2-1)}\begin{pmatrix} t^2 & 0 & -t \\ 0 & t^2-1 & 0 \\ -t & 0 & t^2 \end{pmatrix}$ for $t \neq -1,0,1$, the solutions are $(x,y,z) = \left(\dfrac{t}{t+1}, 0, \dfrac{t}{t+1}\right)$ for $t \neq -1,0,1$

Problem 4.

a) $A^2 + AB + BA + B^2$ b) $A^T A$ c) $3AB+4BA$ d) $A^{-1}BA$ e) BA^2B f) 0

Problem 5.

a) -10 b) 54 c) 40 d) -20

Problem 6.

a) $(7-2y,y,1)$ where y is free

b) $-32a^2+140a-152$, $a=2$ or $a=19/8$

c) $\dfrac{1}{20}\begin{pmatrix}-8 & 8 & -4\\ 36 & 4 & -12\\ -20 & -5 & 10\end{pmatrix}$

d) $(A^{-1})^7\cdot\mathbf{b}$

Problem 7.

a) $\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$

b) $\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$

c) $\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

d) $\begin{pmatrix}1 & 0 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Problem 8.

a) $A^{-1}=\frac{1}{3}\begin{pmatrix}-1 & 2\\ 2 & -1\end{pmatrix}$

b) $A^{-1}=\frac{1}{5}\begin{pmatrix}2 & 6 & -3\\ 1 & -2 & 1\\ -5 & -5 & 5\end{pmatrix}$

c) A not invertible

Problem 9.

a) $(x,y,z)=(2,0,-1)$

b) $|A|=-a(2a+3)$, and $|A|=0$ for $a=0$ and $a=-3/2$

c) $a=0$

d) $\begin{pmatrix}0 & 3 & 1\\ 3 & 8 & -2\\ 1 & -2 & 10\end{pmatrix}$