EBA1180 Mathematics for Data Science
autumn 2022
Exercises

> *... if I couldn't formulate a problem in economic theory mathematically, I didn't know what I was doing.*
>
> R. Lucas

Lecture 4 and 5

Sec. 11.4.1-3, 6.11.4, 11.2, 11.3.2, 11.4.4-10, 4.9.2, 6.11.4, 11.2, 11.3.2: Geometric series. Annuities. Euler's number and continuous compounding of interest.

Here are recommended exercises from the textbook [SHSC].

Section **11.2** exercise 1-3, 5
Section **4.9** exercise 1, 2, 4, 5
Section **11.3** exercise 1b, 2b
Section **11.4** exercise 1-, 6, 7
Section **11.5** exercise 6
Section **11.6** exercise 3

Problems for the exercise session
Wednesday 7 Sept. from 12 in D2-065

Problem 1 Calculate the sum of the series.

a) $1 + 1.04 + 1.04^2 + 1.04^3 + \cdots + 1.04^{10}$.
b) $1 + 1.04 + 1.04^2 + 1.04^3 + \cdots + 1.04^{20}$.
c) $1 + 1.04 + 1.04^2 + 1.04^3 + \cdots + 1.04^n$.
d) $30\,000 \cdot 1.04^{20} + 30\,000 \cdot 1.04^{19} + 30\,000 \cdot 1.04^{18} + \cdots + 30\,000 \cdot 1.04^2 + 30\,000 \cdot 1.04$.
e) Describe a financial situation where the sum in (d) is used.
f) $1 + \frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \cdots + \frac{1}{1.04^{20}}$.
g) Explain why 1.04^{20} multiplied with the sum in (f) gives the sum in (b).
h) $1 + \frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \cdots + \frac{1}{1.04^n}$.
i) $\frac{30\,000}{1.04} + \frac{30\,000}{1.04^2} + \frac{30\,000}{1.04^3} + \cdots + \frac{30\,000}{1.04^{20}}$.
j) Describe a financial situation where the sum in (i) is used.

Problem 2 Suppose you are paid 500 000 every year for n years with the first payment in one year from now. Assume the interest is 3.5%.

a) Write down the geometric series which gives the present value of the cash flow.
b) Use the geometric series to compute the present value of the cash flow for $n = 10$, $n = 20$, $n = 40$, $n = 80$ and $n = 1000$.
c) Compute the present value of the cash flow if it continues forever.

Problem 3 The nominal annual interest is 4.8%.

a) Assume annual compounding. Determine the annual rate of change (growth factor). Determine the rate of change for 10 years. Determine the effective interest for 10 years.
b) Assume quarterly compounding. Determine the annual rate of change and the effective interest. Determine the rate of change for 10 years. Determine the effective interest for 10 years.
c) Assume monthly compounding. Determine the annual rate of change and the effective interest. Determine the rate of change for 10 years. Determine the effective interest for 10 years.

d) Assume daily compounding. Determine the annual rate of change and the effective interest. Determine the rate of change for 10 years. Determine the effective interest for 10 years.
e) Assume continuous compounding. Determine the annual rate of change and the effective interest. Determine the rate of change for 10 years. Determine the effective interest for 10 years.

Problem 4 You deposit 30 000 into an account with 2.9% nominal interest.
a) Assume annual compounding.
   i) Compute the balance after 10 years.
   ii) Determine the rate of change and the relative change for the 10 years.
b) Assume continuous compounding.
   i) Compute the balance after 10 years.
   ii) Determine the rate of change and the relative change for the 10 years.
   iii) Determine the (annual) effective interest.

Problem 5 You consider an investment of 2 million in an asset which can be sold for 5 million after 20 years.
a) With annual compounding, compute the internal rate of return.
b) With quaterly compounding, compute the internal rate of return.
c) With monthly compounding, compute the internal rate of return.
d) With continuous compounding, compute the internal rate of return. (Hint: Try different rates.)

Problem 6 Hege considers a mortgage with 25 annual payments. She believes she will be able to pay 120 000 each year. First payment is one year from now.
a) Assume the interest is 2.0% with annual compounding. Determine the geometric series which gives the present value of the cash flow. Use it to calculate how much Hege can borrow.
b) Assume the interest is 2.0% with continuous compounding. Determine the geometric series which gives the present value of the cash flow. Use it to calculate how much Hege can borrow.
c) Compare the answers in (a) and (b).

Problem 7 Suppose you will be paid 300 000 every year in n years with the first payment a year from now. Suppose the interest is 3.5% with continuous compounding.
a) Write down the geometric series which gives the present value of the cash flow.
b) Use the geometric series to compute the present value of the cash flow for $n = 10$, $n = 20$, $n = 40$, $n = 80$ og $n = 1000$.
c) Use the geometric series to compute the present value of the cash flow if it continues forever.

Problem 8 Suppose a constant amount $A = 40\,000$ (the annuity) is paid every year in n years with the first payment one year from now. Suppose the nominal interest is r with continuous compounding.
a) Write down the geometric series which gives the present value of the cash flow if $n = 25$ og $r = 2.6\%$. Use this series to calculate the present value.
b) Assume the annuity is paid forever. Write down the infinite geometric series which gives the present value of the cash flow if $r = 2.6\%$. Use this series to calculate the present value.
c) Assume the annuity is paid forever. Determine the interest r such that the present value (K_0) becomes 3 million. (Hint: Try different rates.)
d) Explain why (c) gives the equation

$$e^r = \frac{K_0 + A}{K_0} = \frac{3\,000\,000 + 40\,000}{3\,000\,000} = 1.0133$$

Answers

Problem 1

a) $\frac{1.04^{11}-1}{0.04} = 13.49$.

b) $\frac{1.04^{21}-1}{0.04} = 31.97$.

c) $\frac{1.04^{n+1}-1}{0.04}$.

d) $30\,000 \cdot 1.04 \cdot \frac{1.04^{20}-1}{0.04} = 929\,076.05$.

e) A deposit of 30 000 every year for 20 years (starting today) into an account with 4% interest and annual compounding will give the sum as future value after 20 years.

f) We read the geometric series backwards: $\frac{1}{1.04^{20}} \cdot \frac{1.04^{21}-1}{0.04} = 14.59$.

g) $(1 + \frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \cdots + \frac{1}{1.04^{20}}) \cdot 1.04^{20} = 1.04^{20} + 1.04^{19} + \cdots + 1.04^2 + 1.04 + 1$.

h) $\frac{1}{1.04^n} \cdot \frac{1.04^{n+1}-1}{0.04}$.

i) $\frac{30\,000}{1.04^{20}} \cdot \frac{1.04^{20}-1}{0.04} = 407\,709.79$.

j) The sum represents the present value (what you can borrow) for a 30 000 annuity (starting a year form now) with 4% interest and yearly compounding running for 20 years.

Problem 2

a) $\frac{500\,000}{1.035} + \frac{500\,000}{1.035^2} + \frac{500\,000}{1.035^3} + \cdots + \frac{500\,000}{1.035^n}$.

b) $n = 10 : \frac{500\,000}{1.035^{10}} \cdot \frac{1.035^{10}-1}{0.035} = 4\,158\,302.66$, $n = 20 : \frac{500\,000}{1.035^{20}} \cdot \frac{1.035^{20}-1}{0.035} = 7\,106\,201.65$,
$n = 40 : \frac{500\,000}{1.035^{40}} \cdot \frac{1.035^{40}-1}{0.035} = 10\,677\,536.17$, $n = 80 : \frac{500\,000}{1.035^{80}} \cdot \frac{1.035^{80}-1}{0.035} = 13\,374\,387.83$ and
$n = 1000 : \frac{500\,000}{1.035^{1000}} \cdot \frac{1.035^{1000}-1}{0.035} = 14\,285\,714.29$.

c) $\frac{500\,000}{1.035^n} \cdot \frac{1.035^n-1}{0.035} = 500\,000 \cdot \frac{1-\frac{1}{1.035^n}}{0.035}$ which approaches $500\,000 \cdot \frac{1}{0.035} = 14\,285\,714.29$ more and more as n becomes bigger and bigger ("n approches infinity", often written "$n \to \infty$").

Problem 3

a) Annual rate of change: 1.048, rate of change for 10 years: $1.048^{10} = 1.5981$, effective interest for 10 years: 59.81%.

b) Annual rate of change: 1.0489, rate of change for 10 years: 1.6115, effective interest for 10 years: 61.15%.

c) Annual rate of change: 1.0491, rate of change for 10 years: 1.6145, effective interest for 10 years: 61.45%.

d) Annual rate of change: 1.0492, rate of change for 10 years: 1.6160, effective interest for 10 years: 61.60%.

e) Annual rate of change: 1.0492, rate of change for 10 years: 1.6161, effective interest for 10 years: 61.61%.

Problem 4

a) i) 39 927.77
   ii) rate of change: 1.3309, relative change: 33.09%

b) i) 40 092.82
   ii) rate of change: 1.3364, relative change: 33.64%
   iii) 2.94%

Problem 5

a) $2.5^{\frac{1}{20}} - 1 = 4.69\%$

b) 4.61%

c) 4.59%

d) Obtain the equation $e^r = 2.5^{\frac{1}{20}} = 1.0469$ and try: $r = 4.58\%$.

Problem 6

a) Present value:
$120\,000 \cdot \frac{1}{1.02} + 120\,000 \cdot \frac{1}{1.02^2} + 120\,000 \cdot \frac{1}{1.02^3} + \cdots + 120\,000 \cdot \frac{1}{1.02^{24}} + 120\,000 \cdot \frac{1}{1.02^{25}}$.
Mortgage: 2 342 814.78

b) Present value:
$120\,000 \cdot \frac{1}{e^{0.02}} + 120\,000 \cdot \frac{1}{(e^{0.02})^2} + 120\,000 \cdot \frac{1}{(e^{0.02})^3} + \cdots + 120\,000 \cdot \frac{1}{(e^{0.02})^{24}} + 120\,000 \cdot \frac{1}{(e^{0.02})^{25}}$.
Mortgage: $120\,000 \cdot \frac{1}{e^{0.02\cdot 25}} \cdot \frac{e^{0.02\cdot 25}-1}{e^{0.02}-1} = 2\,337\,286.57$

c) With continuous compounding Hege can borrow slight less since then the effective interest she has to pay is slightly higher.

Problem 7

a)

$$300\,000 \cdot \frac{1}{e^{0.035}} + 300\,000 \cdot \frac{1}{(e^{0.035})^2} + \cdots + 300\,000 \cdot \frac{1}{(e^{0.035})^{n-1}} + 300\,000 \cdot \frac{1}{(e^{0.035})^n}$$

b) The sum of the geometric series: $300\,000 \cdot \frac{1}{(e^{0.035})^n} \cdot \frac{(e^{0.035})^n-1}{e^{0.035}-1}$. For $n = 10$: $2\,487\,206.55$ for $n = 20$: $4\,239\,911.38$ for $n = 40$: $6\,345\,389.07$ for $n = 80$: $7\,910\,142.75$ for $n = 1000$: $8\,422\,303.55$.

c) $300\,000 \cdot \frac{1}{(e^{0.035})^n} \cdot \frac{(e^{0.035})^n-1}{e^{0.035}-1} = 300\,000 \cdot \frac{1-(e^{0.035})^{-n}}{e^{0.035}-1}$ approaches $300\,000 \cdot \frac{1}{e^{0.035}-1} = 8\,422\,303.55$ when n grows bigger and bigger.

Problem 8

a)

$$\begin{aligned}&40\,000 \cdot \frac{1}{e^{0.026}} + 40\,000 \cdot \frac{1}{(e^{0.026})^2} + \cdots + 40\,000 \cdot \frac{1}{(e^{0.026})^{24}} + 40\,000 \cdot \frac{1}{(e^{0.026})^{25}}\\ &= 40\,000 \cdot \frac{1}{(e^{0.026})^{25}} \cdot \frac{(e^{0.026})^{25}-1}{e^{0.026}-1} = 40\,000 \cdot \frac{1}{e^{0.026\cdot 25}} \cdot \frac{e^{0.026\cdot 25}-1}{e^{0.026}-1} = 725\,796.53\end{aligned}$$

b)

$$\begin{aligned}&40\,000 \cdot \frac{1}{e^{0.026}} + 40\,000 \cdot \frac{1}{(e^{0.026})^2} + \cdots + 40\,000 \cdot \frac{1}{(e^{0.026})^n} + \dots\\ &= 40\,000 \cdot \frac{1}{e^{0.026}-1} = 1\,518\,548.20\end{aligned}$$

c) Obtain the equation $e^r = 1.0133$ and try: $r = 1.32\%$.