EBA1180, lecture 15, 17 Oct. 2023, Runar Ile

Plan: Repetition

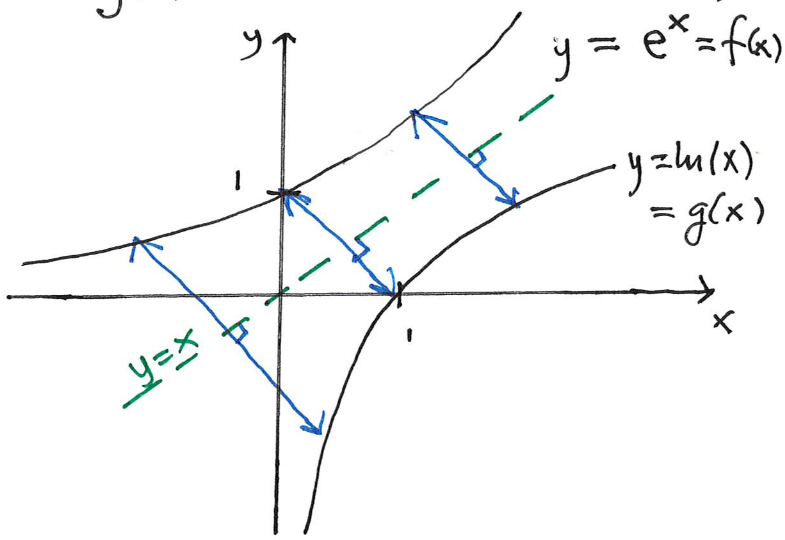
1. Inverse functions
2. Logaritmic and exponential functions
3. Asymptotes

1. Inverse functions

Definition:

$f(g(x)) = x$ for all x in D_g

$g(f(x)) = x$ — " — D_f



*) The graphs are symmetric around the line $y=x$

*) For $f(x)$ to have an inverse function $f(x)$ has to be strictly increasing or strictly decreasing

*) $D_g = R_f$ and $R_g = D_f$

How to find $g(x)$ and D_g in practice

Prob 5d $\quad f(x) = 20 + \dfrac{1}{x-3}, \quad D_f = \langle 3, \rightarrow \rangle$

We find $g(x)$ and D_g.

① Solve the eq. $y = f(x)$ for x

that is $\quad y = 20 + \dfrac{1}{x-3} \quad \Big| \cdot (x-3)$

$$y(x-3) = 20(x-3) + 1$$

$$yx - 3y = 20x - 60 + 1 = 20x - 59$$

$$yx - 20x = 3y - 59$$

$$(y-20)x = 3y - 59 \quad \big| : (y-20)$$

$$x = \frac{3y-59}{y-20}$$

$$x \overset{\text{poly. div.}}{=} 3 + \frac{1}{y-20}$$

② Exchanges variables $(y \leftrightarrow x)$

$$\underline{\underline{y = g(x) = 3 + \frac{1}{x-20}}}$$

③ Put $D_g = R_f$ and find R_f.

R_f = the set of function values of $f(x)$ for $x \in D_f$.

Note that $f(x) \xrightarrow[x \to 3^+]{} +\infty$ and

$f(x) \xrightarrow[x \to \infty]{} 20^+$ so $D_g = R_f = \underline{\underline{\langle 20, \rightarrow \rangle}}$

alt.: $20 < x$

alt.: $x > 20$

2. Logaritmic and exponential functions

Probl. 6 Given $\ln(2) = 0.6931$, $\ln(3) = 1.0986$ and $\ln(5) = 1.6094$. Then (without calc.)

d) $\ln \frac{1000000}{27} = \ln(10^6) - \ln(3^3)$

$= 6 \cdot \ln(10) - 3\ln(3)$

$= 6(\ln(2) + \ln(5)) - 3\ln(3)$

$= 6(0.6931 + 1.6094) - 3 \cdot 1.0986$

$= \underline{\underline{10.5192}}$

f) $\ln(\sqrt[10]{6}) = \ln(6^{\frac{1}{10}}) = \frac{\ln(6)}{10} = \frac{\ln(2) + \ln(3)}{10}$

$= \underline{\underline{0.1792}}$

$f(x) = a^x$, D_f = all numbers on the number line

$a > 0, a \neq 1$

$g(x) = \log_a(x)$, $D_g = \langle 0, \rightarrow \rangle = R_f$.

Ex How long time will it take to double the deposit on an account with 3% interest?

Solution $f(x) = 1.03^x$ is the balance after x years if the deposit was 1. We have to solve the equation $1.03^x = 2$

then $x = \log_{1.03}(2)$

But, we cannot put this into the calculator directly.
Instead we put the LHS and RHS into $\ln(x) = \log_e(x)$.

$$\ln(1.03^x) = \ln(2)$$

$$x \cdot \ln(1.03) = \ln(2) \quad | : \ln(1.03)$$

$$x = \frac{\ln(2)}{\ln(1.03)} \approx \underline{\underline{23.45}}.$$

This also means that

$$\log_{1.03}(2) = \frac{\ln(2)}{\ln(1.03)}$$

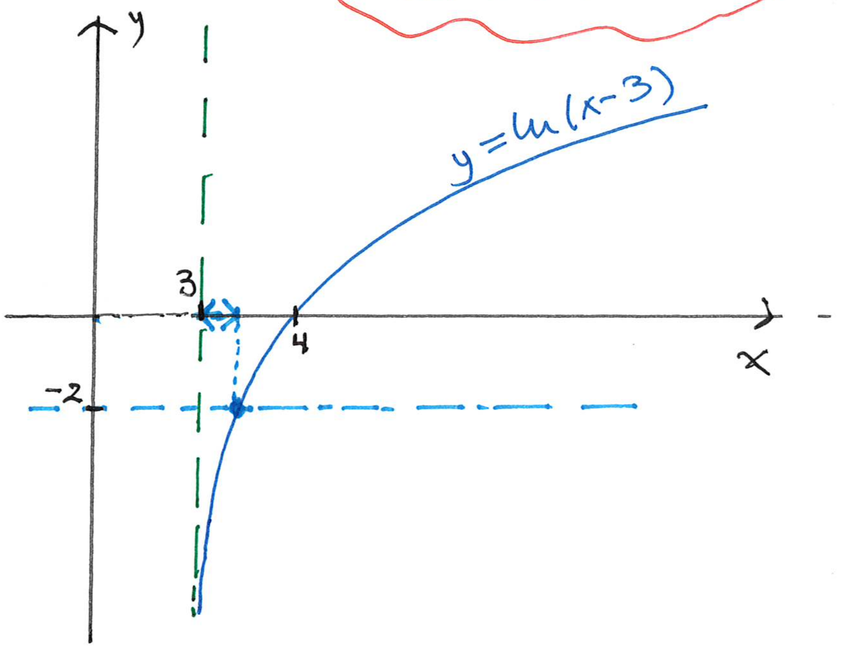
Pattern: $$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

<u>Problem 8c</u> $\ln(x-3) < -2$

Start: 11.02

Since e^x is strictly <u>increasing</u> we can put the LHS and RHS into e^x and get an equivalent inequality:

$$e^{\ln(x-3)} < e^{-2}$$



$$x - 3 < e^{-2}$$

$$x < 3 + e^{-2}$$. But note

that the original inequality is only defined for $x > 3$. So the set of solutions is $\underline{\underline{\langle 3, 3+e^{-2} \rangle}}$

Probl 8e

$$\frac{3e^x}{e^x+1} < 5$$

We could put $u = e^x$ and solve

$$\frac{3u}{u+1} < 5$$

$$\frac{3u}{u+1} - 5 < 0$$

one fraction and sign. diag.

But here it is simpler to multiply each side with $e^x + 1$. Because $e^x + 1$ is greater than 0 whatever x is, this doesn't change the inequality.

$$3e^x < 5(e^x + 1) = 5e^x + 5$$

$$-5 < 2e^x \qquad | : 2$$

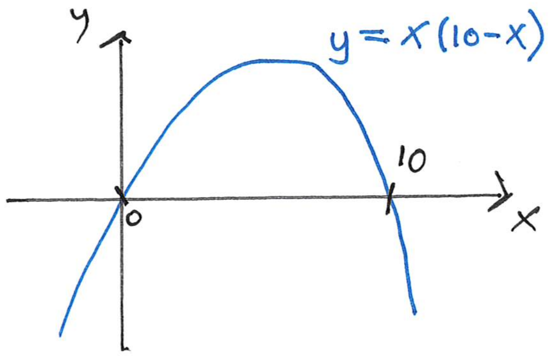
$$-\frac{5}{2} < e^x$$

and this is true for all values of x.

3. Asymptots

Problem 9 Determine the asymptots of $f(x)$

b) $f(x) = e^{x(10-x)} + 50$



We note that $x(10-x) \xrightarrow[x \to \pm\infty]{} -\infty$

Hence $e^{x(10-x)} \xrightarrow[x \to \pm\infty]{} 0^+$

and so $f(x) \xrightarrow[x \to \pm\infty]{} 50^+$ and $\underline{\underline{y = 50}}$ is a horizontal asymptote. (both for $x \to \infty$ and $x \to -\infty$)

d) $f(x) = \ln(10-x)$
is only defined for $x < 10$.
When $x \to 10^-$ then $10 - x \to 0^+$ and

$$f(x) = \ln(10-x) \xrightarrow[x \to 10^-]{} -\infty$$

So $\underline{\underline{x = 10}}$ is a vertical asymptote for $f(x)$.

Probl. 9 f) $f(x) = \ln(\overbrace{120x+10}^{\text{pos. in } D_f}) - \ln(\overbrace{20x-30}^{\text{pos. in } D_f})$

$D_f = \langle \frac{3}{2}, \rightarrow \rangle$

$$= \ln\left(\frac{120x+10}{20x-30}\right)$$

Note that $\frac{120x+10}{20x-30} \xrightarrow[x\to\infty]{} \frac{120}{20} = 6$

so $f(x) \xrightarrow[x\to\infty]{} \ln(6)$ and $\underline{\underline{y = \ln(6)}}$ is a horizontal asymptote

Note also $\frac{120x+10}{20x-30} \xrightarrow[x\to\frac{3}{2}^+]{} +\infty$

Then $f(x) \xrightarrow[x\to\frac{3}{2}^+]{} +\infty$ so $\underline{\underline{x = \frac{3}{2}}}$ is a vertical asymptote.

Probl 10 Find inverse function $g(x)$ and D_g.

c) $f(x) = e^{\frac{2}{x+10}}$, $D_f = [0, \rightarrow \rangle$

① solve the eq. $e^{\frac{2}{x+10}} = y$ for x.

$$\frac{2}{x+10} = \ln\left(e^{\frac{2}{x+10}}\right) = \ln(y) \quad \Big| \cdot (x+10)$$

$$2 = \ln(y)\cdot(x+10) = \ln(y)\cdot x + 10\ln(y)$$

$$2 - 10\ln(y) = \ln(y)\cdot x \quad \Big| : \ln(y)$$

$$x = \frac{2-10\ln(y)}{\ln(y)} = \frac{2}{\ln(y)} - 10$$

(2) $g(x) = \dfrac{2}{\ln(x)} - 10$

(3) $D_g = R_f$. Note that $\dfrac{2}{x+10}$ is a decreasing function. So max. is $\dfrac{2}{0+10} = \dfrac{2}{10} = \dfrac{1}{5}$

And $\dfrac{2}{x+10} \xrightarrow[x \to \infty]{} 0$

Max. of $f(x)$ is $f(0) = e^{\frac{1}{5}}$

and $f(x) \xrightarrow[x \to \infty]{} e^{0} = 1$

so $D_g = R_f = \langle 1, e^{\frac{1}{5}}]$