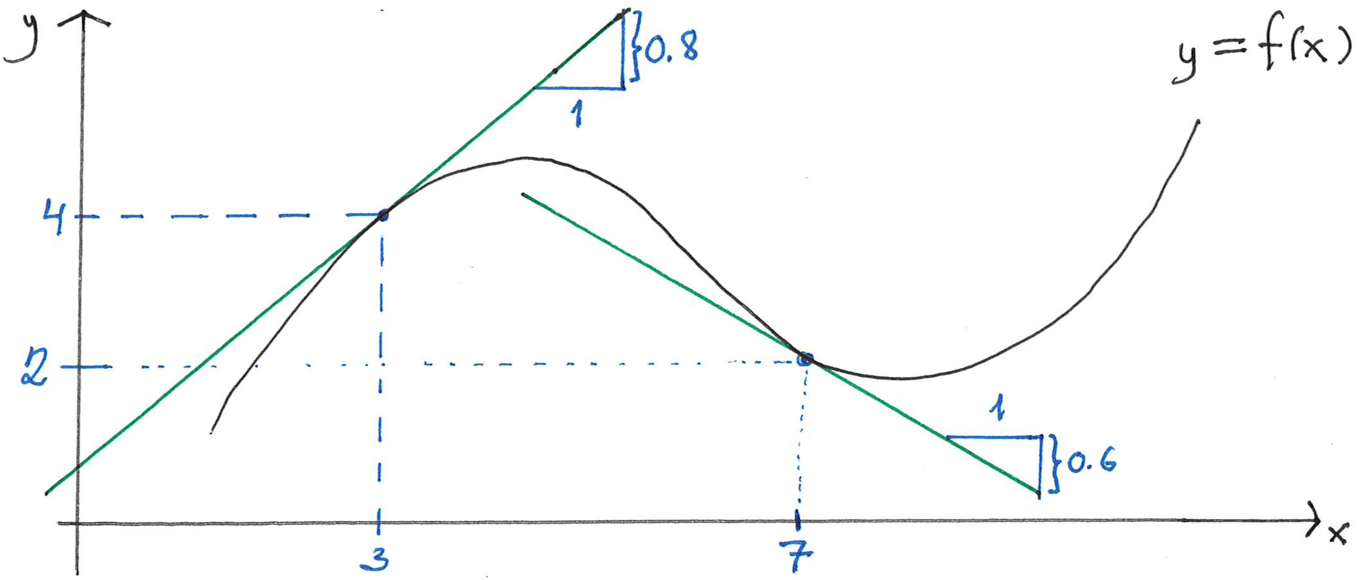
EBA 1180, lecture 16, 18 Oct. 2023, Runar Ile

Plan
1. Tangents and the derivative
2. The derivative as a function
3. Rules for differentiation

1. Tangents and the derivative



- The tangent of the graph of f(x) at the point (3, 4) has slope 0.8
  We write $f'(3) = 0.8$
- The tangent of the graph of f(x) at the point (7, 2) has slope -0.6
  We write $f'(7) = -0.6$

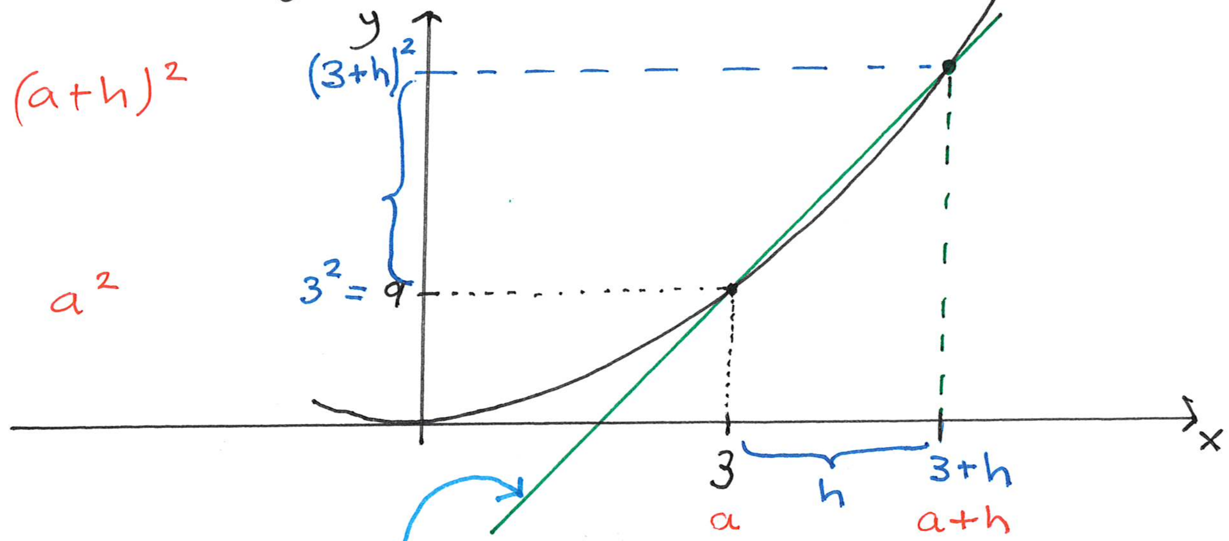
Two important applications

1) To determine where the function f(x) increases / decreases and max/min

2) Approximate complicated functions with linear functions
   - typical for economic models

How to find the slope of the tangent ?

<u>Ex</u> $f(x) = x^2$ and $(3, 9)$. What is the slope of the tangent here ?



The slope of this secant line is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\overset{(a+h)^2 - a^2}{(3+h)^2 - 3^2}}{h} = \frac{\overset{(a+h)(a+h) - a^2}{(3+h)(3+h) - 9}}{h}$$

$$= \frac{\overset{a^2+2ah+h^2-a^2}{9 + 2\cdot 3h + h^2 - 9}}{h} = \frac{\overset{h^2+2ah}{h^2 + 6h}}{h} = \frac{\overset{h(h+2a)}{\cancel{h}(h+6)}}{\cancel{h}}$$

$$h + 2a \xrightarrow[h\to 0]{} 2a$$

$= h + 6 \xrightarrow[h\to 0]{} 6$ which has to be the slope of the tangent line to $f(x)$ through $(3, 9)$.

We write $f'(3) = 6$

also $f'(a) = 2a$

2. The derivative as a function

In the example: If $x = a$ then $f'(a) = 2a$
- this is a function, we use x as variable:

$$f'(x) = 2x$$

E.g. the slope of the tangent of $f(x)$
- at $(-3, 9)$ is $f'(-3) = 2\cdot(-3) = -6$
- at $(1, 1)$ is $f'(1) = 2\cdot 1 = 2$
- at $(10, 100)$ is $f'(10) = 2\cdot 10 = 20$

We could do a similar calculation with $f(x) = x^3$ and get $f'(x) = 3x^2$

---

3. Rules of differentiation

Start: 11.02

Power rule $\quad f(x) = x^n$ gives $f'(x) = n\cdot x^{n-1}$ for all n.

Ex $f(x) = x^{10}$, $f'(x) = 10\cdot x^9$ $\quad (n = 10)$

Ex $f(x) = \sqrt[3]{x}$, $f'(x) = \frac{1}{3} x^{\frac{1}{3}-1}$ $\quad (n = \frac{1}{3})$

$= x^{\frac{1}{3}}$

$$= \frac{1}{3}\cdot x^{-\frac{2}{3}}$$

$$= \frac{1}{3}\cdot \frac{1}{x^{2/3}}$$

$$= \frac{1}{3}\cdot \frac{1}{\sqrt[3]{x^2}} = \frac{1}{3\cdot\sqrt[3]{x^2}}$$

The sum rule If $f(x) = g(x) + h(x)$

then $f'(x) = g'(x) + h'(x)$

Ex $f(x) = x + x^3$ then $f'(x) = 1 + 3x^2$

The constant rule If k is a constant number

and $f(x) = k \cdot g(x)$ then

$$f'(x) = k \cdot g'(x)$$

Ex $k = 7$, $g(x) = x^2$, then $f(x) = 7x^2$

and $f'(x) = 7 \cdot 2x = 14x$

The product rule If $f(x) = g(x) \cdot h(x)$

then $f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$

Ex $f(x) = (5x^3 - 2x + 1) \cdot (3x + 7)$

Calculate $f'(x)$ by using the product rule.

Solution $g(x) = 5x^3 - 2x + 1$ and $h(x) = 3x + 7$

$g'(x) = 15x^2 - 2$ $h'(x) = 3$

so $f'(x) = (15x^2 - 2) \cdot (3x + 7) + (5x^3 - 2x + 1) \cdot (3)$

note the parentheses!

calculate

$$= \underline{\underline{60x^3 + 105x^2 - 12x - 11}}$$

The quotient rule

Suppose $f(x) = \frac{g(x)}{h(x)}$

Then $$f'(x) = \frac{g'(x) \cdot h(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

Ex $f(x) = \frac{3x+1}{2x+5}$. Then

$g(x) = 3x+1$ and $h(x) = 2x+5$

$g'(x) = 3$ $\qquad$ $h'(x) = 2$

$$f'(x) = \frac{3 \cdot (2x+5) - (3x+1) \cdot 2}{(2x+5)^2}$$

note the parentheses!

note this sign! - for the whole parathesis

$$= \frac{6x + 15 - (6x+2)}{(2x+5)^2}$$

note this one!!!

$$= \frac{6x + 15 - 6x - 2}{(2x+5)^2}$$

$$= \frac{13}{(2x+5)^2}$$

usually better to keep the paranthesis in the denominator

The chain rule

If $f(x) = g(u(x))$

the inner function

the outer function

then $f'(x) = g'(u) \cdot u'(x)$ where $u = u(x)$

Ex $f(x) = (x^2+2)^{10}$

Put $u = u(x) = x^2+2$ and $g(u) = u^{10}$

$u'(x) = 2x$ $\qquad g'(u) = 10u^9$

Then $f'(x) = 10u^9 \cdot 2x = 10 \cdot (x^2+2)^9 \cdot 2x$

$= 20x(x^2+2)^9$

Two functions

$f(x) = e^x$ and $g(x) = \ln(x)$

$f'(x) = e^x$ $\qquad g'(x) = \frac{1}{x}$

Ex $f(x) = e^{3x}$

$u = 3x$ and $g(u) = e^u$

$u'(x) = 3$ $\qquad g'(u) = e^u$

$f'(x) = e^u \cdot 3 = 3e^{3x}$

Ex $f(x) = \ln(x^2+1)$

$\left.\begin{array}{ll} u = x^2+1 \ \& & g(u) = \ln(u) \\ u'(x) = 2x & g'(u) = \frac{1}{u} \end{array}\right\}$

$f'(x) = \frac{1}{u} \cdot 2x$

$= \frac{1}{x^2+1} \cdot 2x$

$= \frac{2x}{x^2+1}$