EBA 1180, lecture 21, 7 Nov. 2023, Runar Ile

Plan: 1. Repetition (problems from last week
1d implicit differentiation
2 implicitly defined curves
6b concave / convex functions
8b Convex optimization

1. Repetition

1d) Implicit differentiation

$$x^3 - 3xy + y^2 = 0 \qquad (*)$$

We find an expression for y' in y and x by differentiating each side of $(*)$ with respect to x, and then solve the eq. for y'.

Partial calculations

$$(xy)'_x \overset{\text{prod. rule}}{=} (x)'_x \cdot y + x \cdot y'_x$$
$$= 1 \cdot y + x \cdot y' = y + xy'$$

$$(y^2)'_x \overset{\text{chain rule}}{=} 2y \cdot y'$$

Then $(*)$ gives

$$3x^2 - 3(y + x\underline{y'}) + 2y\underline{y'} = 0$$

Solve this eq. for y':

$$3x^2 - 3y - \underline{3xy'} + \underline{2yy'} = 0$$

$$(2y - 3x)y' = 3(y - x^2)$$

$$y' = \frac{3(y - x^2)}{(2y - 3x)} \qquad (**)$$

Assume $\underline{x=2}$, we find the possible y-values by solving $(*)$ with $x=2$:

$$2^3 - 3\cdot 2\cdot y + y^2 = 0$$
$$y^2 - 6y = -8$$
$$(y-3)^2 = -8+9 = 1$$

so either $y-3=1$ or $y-3=-1$

that is $\underline{\underline{y=4}}$, $\underline{\underline{y=2}}$

We use the point-slope formula to find the two tangent functions through the points $(2,4)$ and $(2,2)$.

$(2,4)$ $\quad y' = \dfrac{3\cdot(4-2^2)}{2\cdot 4 - 3\cdot 2} = 0$

so the tangentfunction is constant: $\underline{\underline{h_1(x) = 4}}$

$(2,2)$ $\quad y' = \dfrac{3\cdot(2-2^2)}{2\cdot 2 - 3\cdot 2} = \dfrac{3(-2)}{-2} = 3$

so the point-slope formula gives

$$h_2(x) - 2 = 3(x-2)$$

that is $\underline{\underline{h_2(x) = 3x - 4}}$

MET1180 Matematikk for siviløkonomer
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Oppgaver

I came to the position that mathematical analysis is not one of the many ways of doing economic theory: it is the only way.

R. Lucas

Forelesning 19 – 20

Kap 4.5, 4.7: Implisitt derivasjon. Den andrederiverte og konvekse/konkave funksjoner.

[L] **4.5** 1-3
[L] **4.7** 1-11

Flervalgseksamen 2016v oppg 15
Flervalgseksamen 2016h oppg 11
Flervalgseksamen 2017v oppg 11

Oppgaver for veiledningstimene torsdag 2/11 fra 10 i D1-065/70

Oppgave 1 Uttrykk y' ved hjelp av y og x ved implisitt derivasjon. Finn alle løsninger for y gitt at $x = a$ og bestem funksjonsuttrykkene for tangentene i disse punktene.

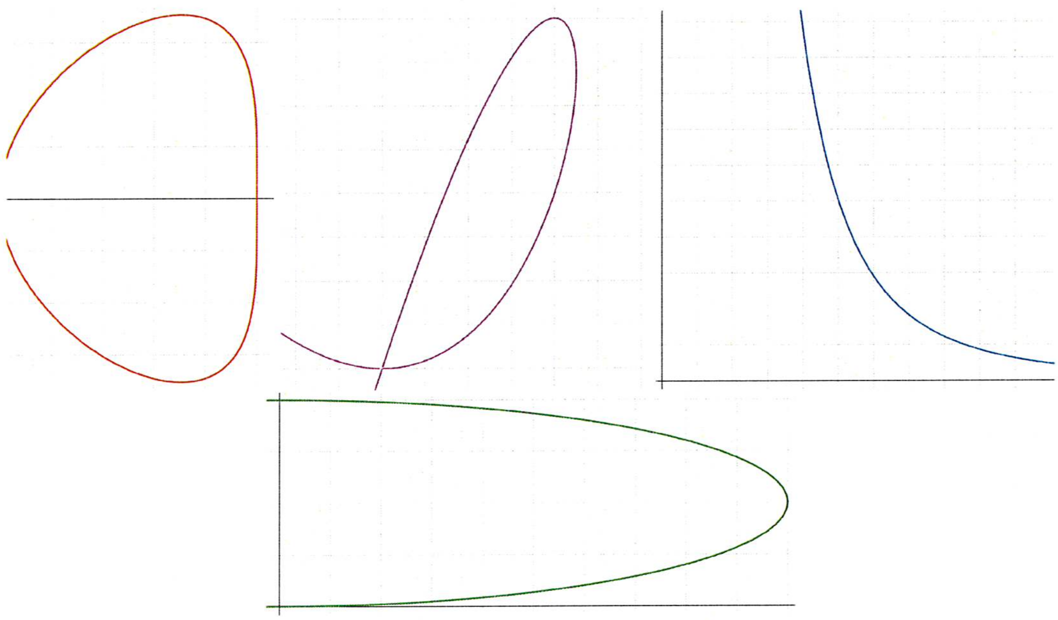
a) $x^2 + 25y^2 - 50y = 0$ og $a = 4$

b) $x^{3,27} y^{1,09} = 1$ og $a = 1$

c) $x^4 - x^2 + y^4 = 0$ og $a = \frac{\sqrt{2}}{2}$

d) $x^3 - 3xy + y^2 = 0$ og $a = 2$

Oppgave 2 I figur 1 ser du grafene til de implisitt definerte kurvene i oppgave 1. Finn kurvene og likningene som hører sammen. Tegn også inn tangentene fra oppgave 1.



Figur 1: Fire implisitt definerte kurver

2) Ellimination is the strategy.

① In 1a, c and d we got <u>two</u> y-values for one x-value. So 1a, c and d cannot be the blue graph (to the right)

So <u>1b has to be the blue graph</u>

② The red and the green graphs are symmetric and so their tangents are symmetric too. In particular the slopes are only changing sign (for a fixed x-value). This is the case for 1a and c.

So <u>1d has to be the purple one</u> (in the middle)

③ In 1a we have both y-values positive. In 1c one y-value is negative.

If the thicker horizontal lines are the x-axes, then

1a has to be the green (bottom) one

1c —— " —— red (upper left) one

6b) $f(x) = \ln(x^2-2x+2) - \frac{x}{4} + 1$

Note $x^2-2x+2 = (x-1)^2+1 \geqslant 1$ so

$f(x)$ is defined on the whole number line.

$$f'(x) = \left[\ln(x^2-2x+2)\right]' - \frac{1}{4} + 0$$

Chain rule with

$u = x^2-2x+2$ and $g(u) = \ln(u)$

$u'(x) = 2x-2$ $g'(u) = \frac{1}{u}$

$$= \underline{\underline{\frac{2x-2}{x^2-2x+2} - \frac{1}{4}}}$$

$$f''(x) = \frac{(2x-2)'\cdot(x^2-2x+2) - (2x-2)\cdot(x^2-2x+2)'}{(x^2-2x+2)^2}$$

$$= \frac{2\cdot(x^2-2x+2) - \overbrace{(2x-2)(2x-2)}^{4x^2-8x+4}}{(x^2-2x+2)^2}$$

$$= \frac{2x^2-4x+4-4x^2+8x-4}{(x^2-2x+2)^2}$$

$$= \underline{\underline{\frac{-2x^2+4x}{(x^2-2x+2)^2}}} = \frac{-2x\cdot(x-2)}{[(x-1)^2+1]^2}$$

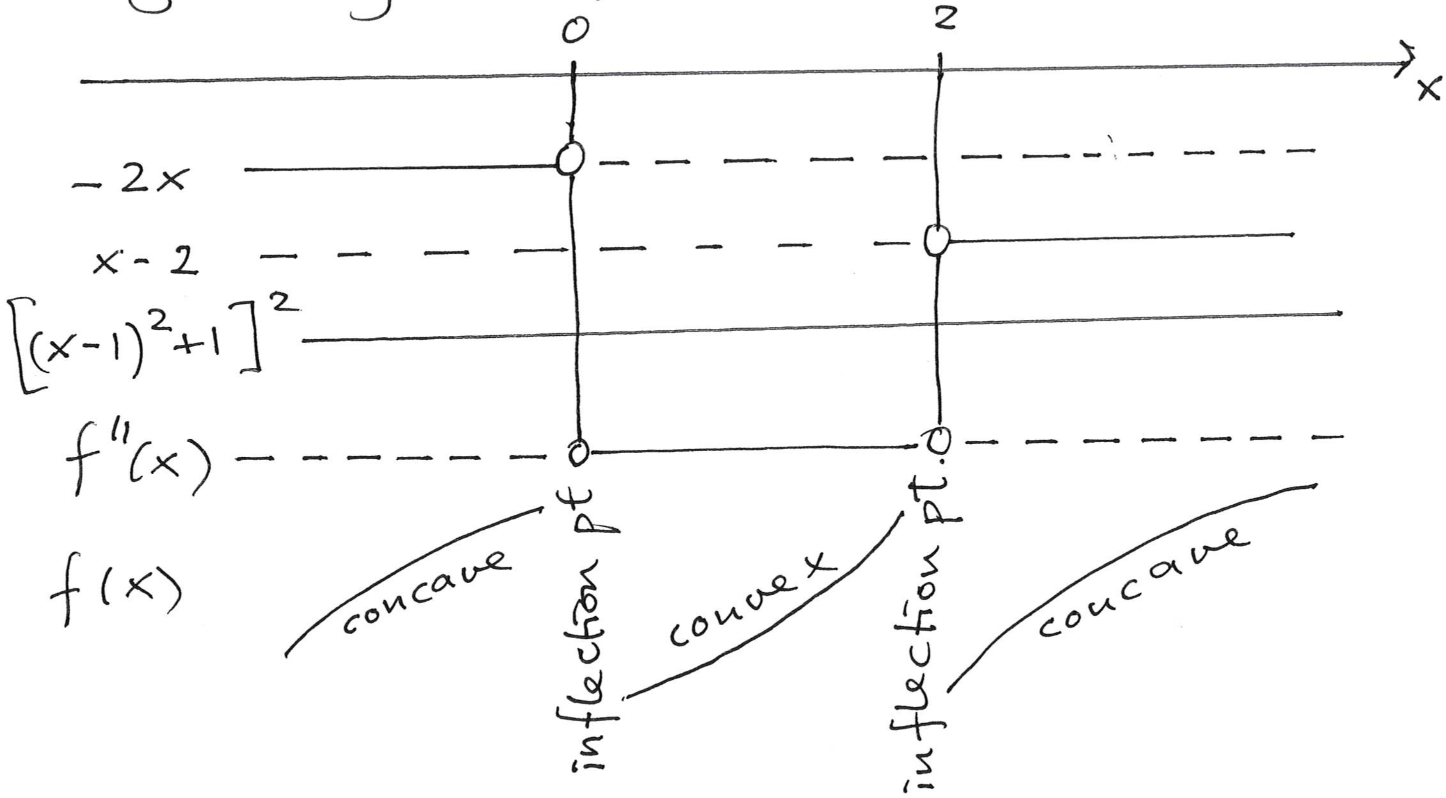
Solve eq. $f''(x) = 0$

that is $-2x \cdot (x-2) = 0$

so $-2x = 0$ or $x - 2 = 0$

that is $\underline{\underline{x = 0}}$ or $\underline{\underline{x = 2}}$

Sign diagram for $f''(x)$.



Conclusion

$f(x)$ is concave for x in $\langle \leftarrow, 0]$

— " — convex — " — $[0, 2]$

— " — concave — " — $[2, \rightarrow\rangle$

Then the inflection points for $f(x)$

are $\underline{\underline{x = 0}}$ and $\underline{\underline{x = 2}}$

because $f''(x)$ changes sign here.

Start: 11.10

8b) • Determine the (local) max/min points for $f(x)$.
• Explain why they give global max/min by using convexity.
• Calculate max/min. of $f(x)$.

$$f(x) = \frac{-1}{x(x-6)}, \quad D_f = \langle 0, 6 \rangle.$$

• $f'(x) \overset{\text{frac. rule}}{=} \dfrac{2x-6}{[x(x-6)]^2}$. Stationary point: Solutions of eq. $f'(x) = 0$

that is $2x - 6 = 0$ $\quad (x(x-6) \neq 0)$

$\underline{x = 3}$ $\quad (3\cdot(3-6) \neq 0$ -ok$)$

Numerator of $f'(x)$ is changing sign from $-$ to $+$ at $x = 3$, and denominator is pos.

So $f'(x)$ changes sign from $-$ to $+$ at $x=3$ and $\underline{\underline{x=3}}$ is a (local) minimum point

• Calculate $f''(x) = \left[\dfrac{2x-6}{x^2(x-6)^2}\right]'$

$$\left[x^2\cdot(x-6)^2\right]' = 2x(x-6)^2 + x^2\cdot 2(x-6)\cdot 1$$
$$= 2x(x-6)(x-6+x)$$
$$= 2x(x-6)(2x-6)$$

$$f''(x) = \frac{2\cdot x^2(x-6)^2 - (2x-6)\cdot 2x(x-6)(2x-6)}{x^4\cdot(x-6)^4}$$

$$= \frac{2\cancel{x}\,\cancel{(x-6)}\left[x(x-6) - (2x-6)^2\right]}{x^{\cancel{4}\,3}\cdot(x-6)^{\cancel{4}\,3}}$$

$(-3)(-6)$ $(-3)\cdot 12$

$$= \frac{2\cdot\left[-3x^2 + 18x - 36\right]}{x^3(x-6)^3}$$

$$= \frac{-6\left[x^2 - 6x + 12\right]}{x^3\cdot(x-6)^3} = \frac{-6\left[(x-3)^2 + 3\right]}{x^3(x-6)^3}$$

For $x \in \langle 0, 6\rangle$, then $x^3 > 0$ and
$(x-6)^3 < 0$

So $f''(x) = \frac{\text{neg.}}{\text{neg.}} > 0$ for $x \in D_f = \langle 0, 6\rangle$.

Hence $f(x)$ is convex in the whole domain and $x = 3$ is a global minimum point.

- Minimal value of $f(x)$ is

$$f(3) = \frac{-1}{3(3-6)} = \underline{\underline{\frac{1}{9}}}$$

– no maximal value.

2. l'Hopital's rule

It is about limits of the type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

<u>Notation</u> $\lim_{x\to 5} f(x)$ <u>is</u> the number

which $f(x)$ is approaching when x is approaching 5.

<u>Ex</u> $f(x) = \dfrac{3x-3}{\ln(x)}$. Want to find $\lim_{x\to 1} f(x)$.

$$\left.\begin{array}{ll} \text{Numerator}: & 3x-3 \longrightarrow 3\cdot 1-3=0 \\ \text{Denominator}: & \ln(x) \longrightarrow \ln(1)=0 \end{array}\right\} \tfrac{0}{0}\text{-expr.}$$

Then we can use l'Hôpital's rule

$$\lim_{x\to 1} f(x) \overset{\text{l'Hôp}}{=} \lim_{x\to 1} \frac{(3x+3)'}{[\ln(x)]'} = \lim_{x\to 1} \frac{3}{\left(\frac{1}{x}\right)} = \frac{3}{\left(\frac{1}{1}\right)} = \underline{\underline{3}}$$

Check: $f(1.01) = \dfrac{3\cdot 1.01-3}{\ln(1.01)} = 3.050$

$f(0.99) = \dfrac{3\cdot 0.99-3}{\ln(0.99)} \approx 2.9850$

<u>Note</u> Has to be $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$!

Then differentiate numerator and denominator <u>seperately</u> and try to find the limit of the new fraction.

$\underline{\text{Ex}}$ $\lim\limits_{x \to 0} \dfrac{3x}{e^x - 1}$ $\left(\text{"}\frac{0}{0}\text{"}\right)$

$3x \xrightarrow[x \to 0]{} 0$

$e^x - 1 \xrightarrow[x \to 0]{} 1 - 1 = 0$

$\overset{\text{l'Hôp}}{=} \lim\limits_{x \to 0} \dfrac{3}{e^x} = \dfrac{3}{1} = \underline{\underline{3}}$

$\underline{\text{Ex}}$ $\lim\limits_{x \to \infty} \underbrace{\dfrac{x^2}{e^x}}_{\text{"}\frac{\infty}{\infty}\text{"}} \overset{\text{l'Hôp}}{=} \lim\limits_{x \to \infty} \underbrace{\dfrac{2x}{e^x}}_{\text{"}\frac{\infty}{\infty}\text{"}} \overset{\text{l'Hôp}}{=} \lim\limits_{x \to \infty} \dfrac{2}{e^x} = \underline{\underline{0}}$