Inner product (dot product) of vectors

EB A1180
Sect. 38
Spring 24

Def (Inner product): Let $\vec{v}, \vec{w}$ be n-vectors

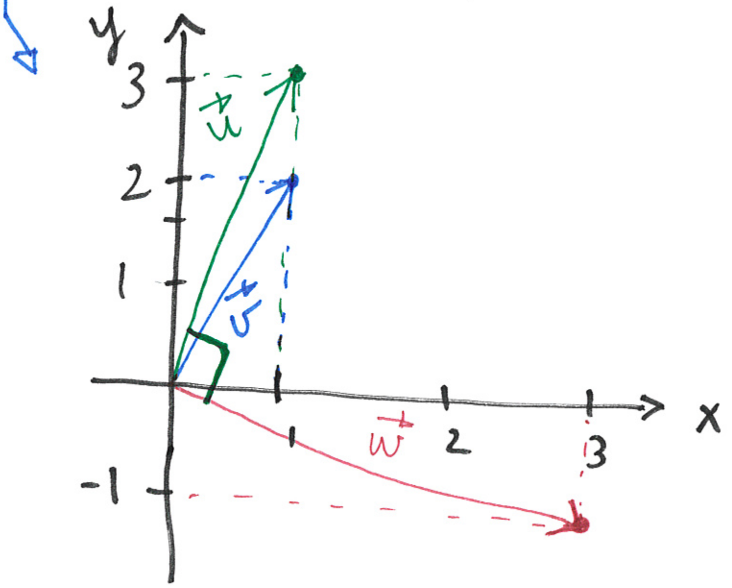
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}. \text{ Then,}$$

$$\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{v} \cdot \vec{w} = 1 \cdot 3 + 2 \cdot (-1) = 3 - 2 = \underline{1}$$

$$\vec{u} \cdot \vec{w} = 3 \cdot 1 + (-1) \cdot 3 = 3 - 3 = \underline{0}$$



NOTE: $\vec{u} \cdot \vec{w} = 0$

and $\vec{u} \perp \vec{w}$

has a 90° angle with

"$\vec{u}$ and $\vec{w}$ are orthogonal"

Result:

$$\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$$

Rules of computation: 1) $\vec{v} \cdot \vec{w} = $ a number

2) $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + \dots + v_n^2 = \|\vec{v}\|^2$

$$\|\vec{v}\|^2 = \left(\sqrt{v_1^2 + v_2^2 + \dots + v_n^2}\right)^2$$
$$= v_1^2 + v_2^2 + \dots + v_n^2$$

3) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

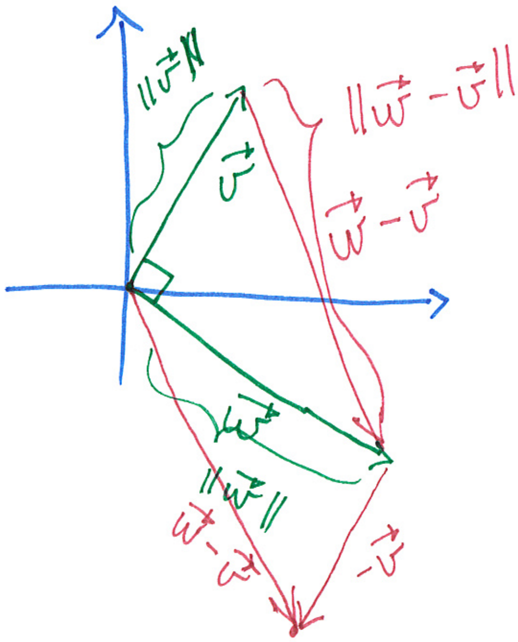
$$(a\vec{u} + b\vec{w}) \cdot \vec{v} = a\,\vec{u} \cdot \vec{v} + b\,\vec{w} \cdot \vec{v}$$

Proof of Result:

Pythagoras

$$\vec{v} \perp \vec{w} \iff \|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{w} - \vec{v}\|^2$$

$$\Updownarrow \quad \text{Rule 2)}$$



$$\underline{(\cancel{v_1^2} + \dots + \cancel{v_n^2}) + (\cancel{w_1^2} + \dots + \cancel{w_n^2})}$$

$$= (w_1 - v_1)^2 + (w_2 - v_2)^2 + \dots + (w_n - v_n)^2$$

$$= \cancel{w_1^2} - 2w_1 v_1 + \cancel{v_1^2} + \dots + \cancel{w_n^2} - 2w_n v_n + \cancel{v_n^2}$$

$$0 = -2w_1v_1 - \dots - 2w_nv_n \quad \Big| : (-2)$$

$$\Updownarrow$$

$$w_1v_1 + \dots + w_nv_n = 0$$

$$\vec{w} \cdot \vec{v} = 0$$

$$\Updownarrow \quad \text{Rule } 3)$$

$$\vec{v} \cdot \vec{w} = 0$$

$$\blacksquare$$

NOTE: $\underbrace{\vec{v} \cdot \vec{w}}_{\text{inner product of n-vectors}} = \underbrace{\vec{v}^T \vec{w}}_{\text{matrix multiplication}}$

Ex: $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$\vec{v} \cdot \vec{w} = 2 \cdot 1 + 1 \cdot (-3) = \underline{-1}$$

SAME! =

$$\vec{v}^T \vec{w} = \underset{1 \times 2}{\begin{bmatrix} 2 & 1 \end{bmatrix}} \underset{2 \times 1}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}} = 2 \cdot 1 + 1 \cdot (-3) = \underline{-1}$$

NB: $\underbrace{\vec{v}\,\vec{w}}_{\text{matrix multiplication}}$ is not defined: $\underset{2 \times 1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \underset{2 \times 1}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}$

NOT SAME

Functions in two variables

Ex: $f(x, y) = 2x + 3y - 1$, linear function

two variables

$f(x, y) = x^2 + y^2$, polynomial function

$f(x, y) = \dfrac{x+y}{x-y}$, rational function

$f(x, y) = x e^y$

General: $f(x, y)$: function expression in x, y

x, y: input variables

$z = f(x, y)$: output variable

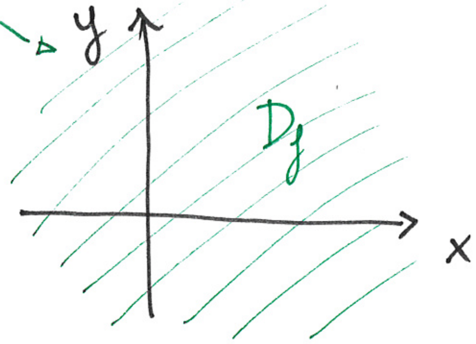
Def (Domain of f):

D_f = domain of f = all coordinate pairs (x, y) that we can input to the function f

Ex: $f(x,y) = 2x + 3y - 1$,

$D_f = \mathbb{R}^2$

Subset of xy-plane; $\mathbb{R}^2$

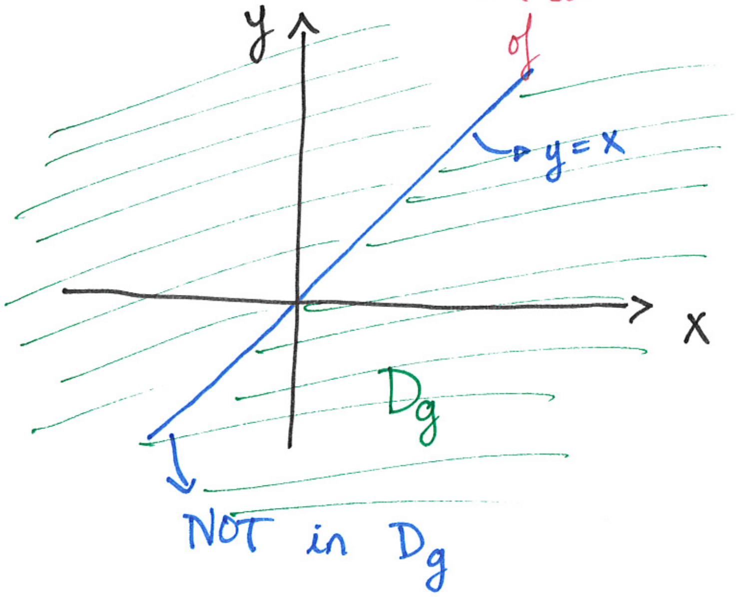


$g(x,y) = \dfrac{x+y}{x-y}$, D_g: $x \neq y$

$x - y \neq 0$; Divide by 0 if $x = y$

$$D_g = \{(x,y) \in \mathbb{R}^2 : x \neq y\}$$

the set of — element in — such that — the set of



$x = y$

$y = x$

Def (Range):

V_f = range of f = all values $f(x,y)$ can attain when $(x,y) \in D_f$

• To find the range, V_f: Find the max/min of f

Ex: $f(x,y) = 2x + 3y - 1$

$D_f = \mathbb{R}^2$

$x \to \infty$, $y = 0 \Rightarrow f(x,y) \to \infty$.

$x \to -\infty$, $y = 0 \Rightarrow f(x,y) \to -\infty$

$$V_f = (-\infty, \infty) = \mathbb{R}$$

$f(x,y) = x^2 + y^2$,

$D_f = \mathbb{R}^2$

$$V_f = [0, \infty)$$

Can only get non-neg. numbers because of squares

Graphs and level curves

Def (Graph of function in two variables)

The graph of a function f in two variables is the set of all points

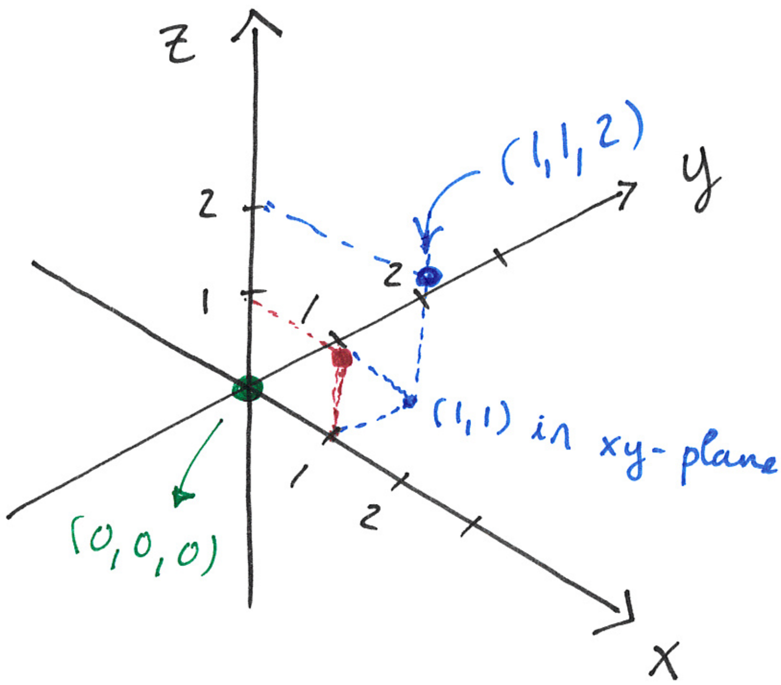
$$(x, y, z)$$

where $(x,y) \in D_f$ and $z = f(x,y)$

- Can draw the graph of f in the xyz-coordinate system.

<u>Ex</u>: $f(x,y) = x^2 + y^2$, $D_f = \mathbb{R}^2$

(x,y)	$(0,0)$	$(1,0)$	$(1,1)$
$z = f(x,y)$	0	1	2
Point in xyz-plane	$(0,0,0)$	$(1,0,1)$	$(1,1,2)$



- The graph of f is called a <u>surface</u>.

<u>Def (Level curves)</u>: All (x,y) such that $f(x,y) = c$ for a constant c.

In general: Graph of a function $f(x,y)$: