EBA1180
Spring 24
Lect. 40

Warm up: $f(x,y) = 2x^2 + xy$

Q1: Stationary points? $\to$ $f'_x = 0$, $f'_y = 0$

Q2: Classify?

Q1: $f'_x = 4x + y = 0 \Rightarrow 0 + y = 0$
$f'_y = x = 0$ $\qquad y = 0$

One stationary point: $(x, y) = \underline{\underline{(0,0)}}$

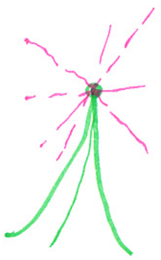
Q2: $H(f) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$

$\det H(f) = 0 - 1 = -1 < 0 \Rightarrow$ Saddle point.

o-o

Stationary points ctd.

Ex: $f(x,y) = \underbrace{x^3 + 3xy + y^3}$, $D_f = \mathbb{R}^2$ $(\mathbb{R} \times \mathbb{R})$



Partial derivatives exist everywhere: No pts. where either f'_x or f'_y are not defined

no boundary points ~~to~~ to consider

Stationary points (also the candidate points):

$$\begin{cases} f'_x = 3x^2 + 3y = 0 \\ f'_y = 3x + 3y^2 = 0 \end{cases}$$

FOC (first order conditions)

$$\begin{cases} x^2 + y = 0 \Rightarrow y = -x^2 \quad : (\star) \\ x + y^2 = 0 \rightarrow x + (-x^2)^2 = 0 \end{cases}$$

$$x + x^4 = 0$$

$$x(1 + x^3) = 0$$

$$x = 0$$

From $(\star)$: $y = -0^2 = 0$

$$1 + x^3 = 0$$

$$x^3 = -1$$

$$x = \sqrt[3]{-1} = -1$$

From $(\star)$: $y = -(-1)^2 = -1$

So the stationary points are $(0, 0)$ and $(-1, -1)$.

Since there are no boundary points or points where either partial derivative is not defined, these are also the candidate points.

The second derivative test

Ex ctd: $f(x,y) = x^3 + 3xy + y^3$

$f'_x = 3x^2 + 3y$

$f'_y = 3x + 3y^2$

Candidate points:

$(x^*, y^*) = (0,0),$
$(-1,-1)$

$f(0,0) = 0^3 + 3 \cdot 0 \cdot 0 + 0^3 = \underline{0}$

$f(-1,-1) = (-1)^3 + 3 \cdot (-1) \cdot (-1) + (-1)^3$

$= -1 + 3 - 1 = \underline{1}$

Classify the candidate points using the second derivative test

Q: Hessian:

$$H(f) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} 6x & 3 \\ 3 & 6y \end{bmatrix}$$

Candidate point $(0,0)$:

$$H(f)(0,0) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \Rightarrow$$

$\det H(f)(0,0)$
$= 0 \cdot 0 - 3 \cdot 3$
$= -9 < 0$

$\Rightarrow$ $(0,0)$ is a saddle point for f.

Candidate point $(-1,-1)$:

$$H(f)(-1,-1) = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}$$

$$\det H(f)(-1,-1) = 36 - 9 = 27 > 0$$

since >0, we also need to compute trace to classify

$$\operatorname{tr} H(f)(-1,-1) = -6 + (-6) = -12 < 0$$

$\Downarrow$ second derivative test

$(-1,-1)$ is a local max. for f.

Global max/min?

Conclusion: $f(x,y) = x^3 + 3xy + y^3$ has no minimum. — " — has a local maximum $f(-1,-1) = 1$.

NOTE: $f(10,10) = 10^3 + 3 \cdot 10 \cdot 10 + 10^3$

$= 2300 > 1$

$1 = f(-1,-1)$: value in loc. max

$\Rightarrow$ $(-1,-1)$ is not a global max. for f.

$\Rightarrow$ f has no global max.

Tangents of level curves

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

Level curve: All (x,y) s.t. $f(x,y) = c$

$$x^2 - 2x + y^2 + 4y = c$$

TRICK: Complete the squares:

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = c + 1 + 4$$

$$(x-1)^2 + (y+2)^2 = c + 5 \quad :(\star)$$

$c+5>0$:

So $c>-5$

$\Rightarrow$ Level curve is a circle with center in $(1,-2)$ and $r = \sqrt{c+5}$.

$c+5=0$:

So $c = -5 \Rightarrow$

Level "curve" is $(1,-2)$

(Need $x-1=0$, $y+2=0$: because sum of squares is $0 \Rightarrow$ Squares must be $0 \Rightarrow x-1=0$, $y+2=0$)

$c+5<0$:

So $c<-5 \Rightarrow$

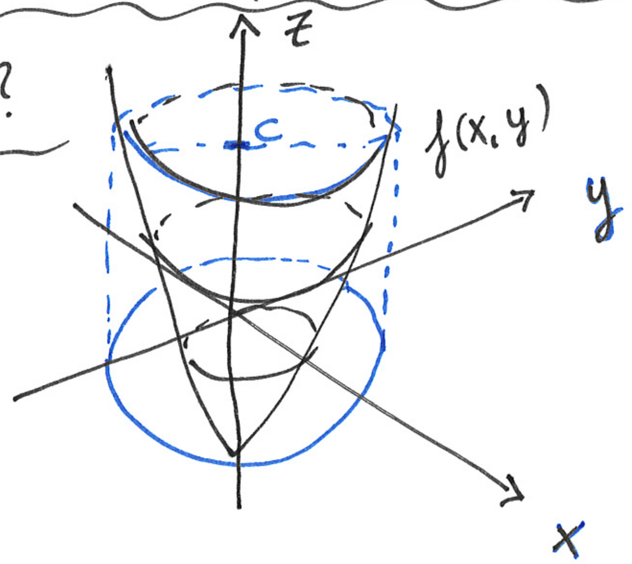
$(\star)$ is never true

$\Rightarrow$ No level curve

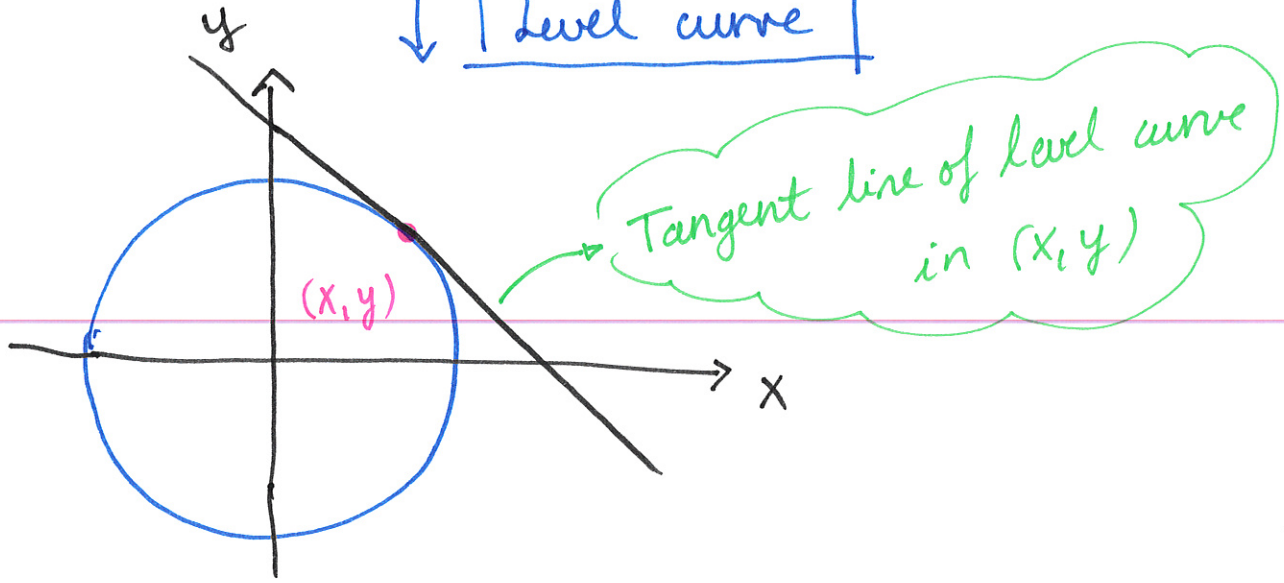
(i.e. f never attains a value $c<-5$)

How to find the tangent line of a level curve in a point (x,y)?

Where are we?



$\downarrow$ $\boxed{\text{Level curve}}$



Ex. Ctd: Tangent line of level curve of f in

$$(x,y) = (-2, 2)?$$

$$f(x,y) = x^2 - 2x + y^2 + 4y$$

1) Find level of level curve corresponding to the point:

$$z = f(-2,2) = (-2)^2 - 2\cdot(-2) + 2^2 + 4\cdot 2 = 20$$

$$\Rightarrow c = 20$$

Level curve at level 20 corresponds to $(x, y) = (-2, 2)$.

2) Point-slope formula:

slope: unknown

$$y - 2 = k\,(x - (-2))$$

$$y = k\,(x+2) + 2 \quad : (\sim)$$

What is the slope k? 3.) Implicit differentiation:

Think $y = y(x)$:

$$\underbrace{(x^2 - 2x + y^2 + 4y)'_x}_{f(x,y)} = \underbrace{(20)'_x}_{C}$$

$$2x - 2 + \underbrace{2y(x)\,y'(x)}_{\text{chain rule}} + 4y'(x) = 0$$

$$2x - 2 + 2y\,y' + 4y' = 0$$

Solve for y': $\quad y' = -\dfrac{2x-2}{2y+4} \overset{\boxed{\text{OBSERVE}}}{=} -\dfrac{f'_x}{f'_y}$

$\frac{dy}{dx}$ = the slope of a lin. func.

$y = ax + b$

$y' = a$; the slope

4.) Insert $(x,y) = (-2,2)$:

$$y'\big|_{(-2,2)} = -\frac{2\cdot(-2)-2}{2\cdot 2+4} = \underbrace{\frac{3}{4}}$$

the slope of the tangent line in $(-2,2)$

Hence, from $(\sim)$:

$$y = \frac{3}{4}(x+2)+2$$

$$= \frac{3}{4}x + \frac{3}{2} + 2$$

$$= \frac{3}{4}x + \frac{7}{2}$$

Result: If $f(x,y) = c$, then

$$f'_x + f'_y\, y' = 0$$

Implicit differentiation + chain rule (same arg. as above)

Hence,

$$\underbrace{y'}_{\text{slope}} = -\frac{f'_x}{f'_y}$$

HOLDS IN GENERAL