Warm-up: $f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}}$

EBA1180
Spring 24
Sect. 41

$f'_x(x,y) = (\sqrt{u})'_x = (u^{\frac{1}{2}})'_x$

Let $u = x^2+y^2$

$= \frac{1}{2} u^{-\frac{1}{2}} \cdot u'_x$

CHAIN RULE

$= \frac{1}{2\sqrt{u}} 2x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$

$= \frac{x}{\sqrt{x^2+y^2}}$

$f'_y(x,y) = \frac{y}{\sqrt{x^2+y^2}}$

By symmetry or repeating same argument

Interpretation of partial derivatives

What does $f'_x(a,b)$ and $f'_y(a,b)$ mean?

Ex: $f(x,y) = x^3 - 3xy + y^3$

$\Rightarrow f'_x(x,y) = 3x^2 - 3y$, $f'_y(x,y) = -3x + 3y^2$

Let $(x,y) = (2,1)$. Then:

$f(2,1) = 2^3 - 3\cdot 2\cdot 1 + 1^3 = 3$

$f'_x(2,1) = 3\cdot 2^2 - 3\cdot 1 = \underline{9}$

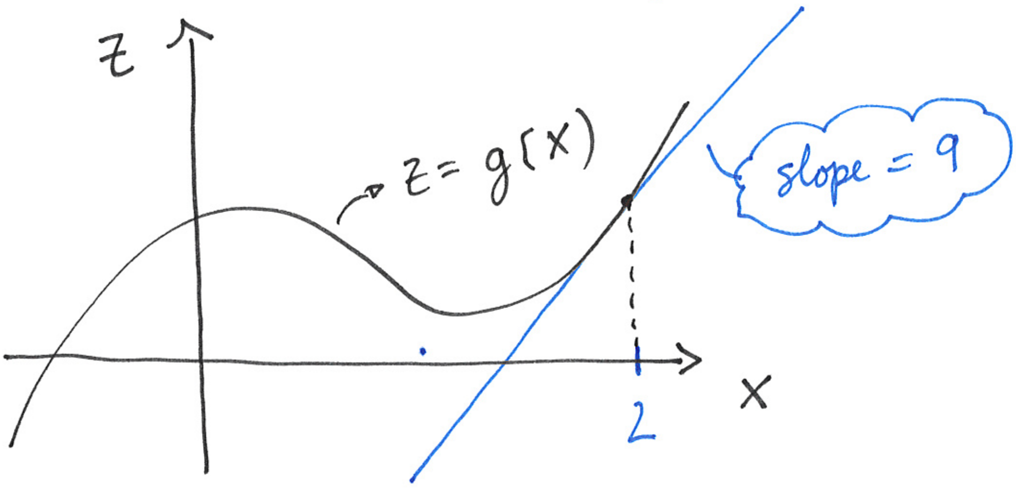
SAME!

$f'_y(2,1) = -3\cdot 2 + 3\cdot 1^2 = \underline{-3}$

In the x-direction $(y=1)$:

$$f(x,1) = x^3 - 3x\cdot 1 + 1^3 = x^3 - 3x + 1 \overset{\text{Defined as}}{=:} g(x)$$

$$\underbrace{g'(x)}_{= f'_x(x,1)} = 3x^2 - 3, \quad \underbrace{g'(2)}_{= f'_x(2,1)} = 3\cdot 2^2 - 3 = \underline{9}$$

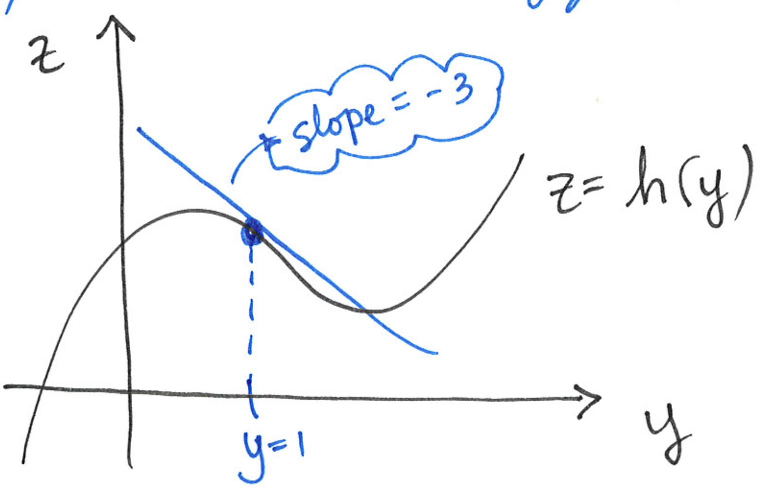


In the y-direction $(x=2)$:

$$f(2,y) = 2^3 - 3\cdot 2\cdot y + y^3 = 8 - 6y + y^3 =: h(y)$$

$$\Rightarrow \underbrace{h'(y)}_{= f'_y(2,y)} = -6 + 3y^2, \text{ so } \underbrace{h'(1)}_{= f'_y(2,1)} = -6 + 3\cdot 1^2 = \underline{-3}$$

NOTE: Same as on pg. 1



Linear approximation of $f(x,y)$ at (x_0,y_0):

Tangent plane of f at (x_0,y_0)

$$L(x,y) = f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

Plane: $ax + by + c$

$f'_x(x_0,y_0)$ $f'_y(x_0,y_0)$

The gradient

Def (gradient): The gradient of $f(x,y)$ is

$$\nabla f = \begin{bmatrix} f'_x \\ f'_y \end{bmatrix}$$

"The gradient of f"   A vector

Ex: $f(x,y) = x^2 - 2x + y^2 + 4y$

Q: $\nabla f = \begin{bmatrix} 2x-2 \\ 2y+4 \end{bmatrix}$

The gradient of f in $(x,y) = (-2,2)$:

Q: $\nabla f(-2,2) = \begin{bmatrix} 2\cdot(-2)-2 \\ 2\cdot 2+4 \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$

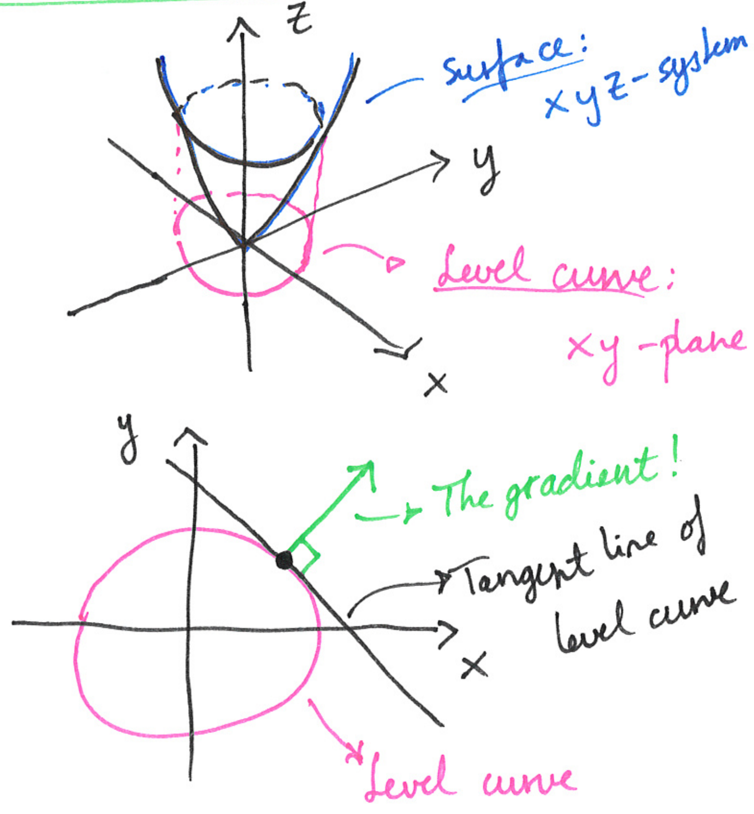
NOTE: The gradient at a point is a normal vector to the tangent line of the level curve at that point.

<u>Ex</u>: $\nabla f(-2,2) = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$

Tangent line of level curve at $(-2,2)$:

$$y = \frac{3}{4}x + \frac{7}{2}$$

From Tuesday lecture!



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{3}{4}x + \frac{7}{2} \end{bmatrix}$$

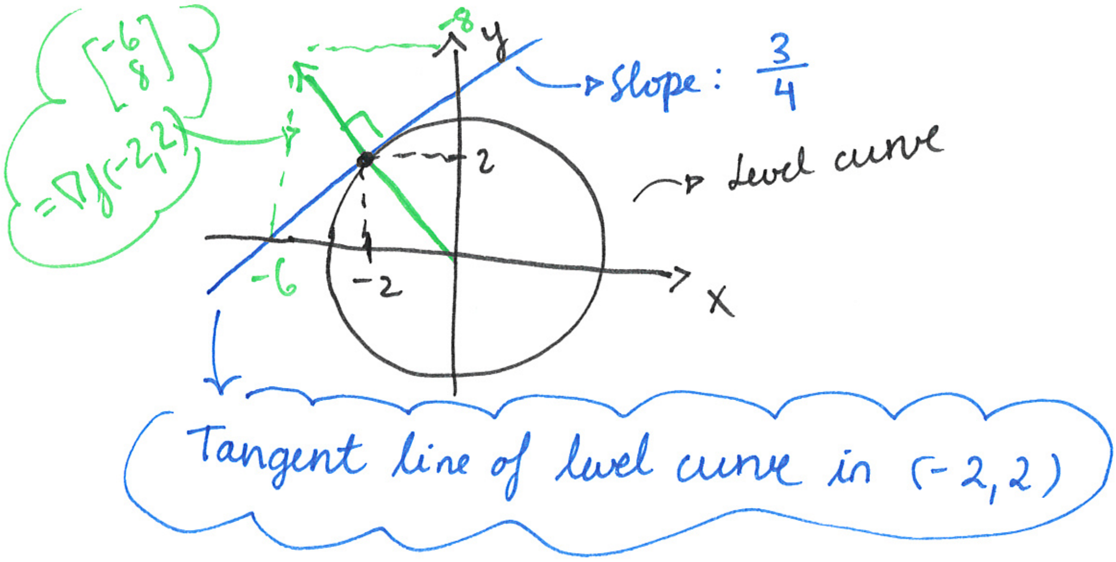
$$= \underbrace{\begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}}_{\text{direction of the tangent line of the level curve}} x + \begin{bmatrix} 0 \\ \frac{7}{2} \end{bmatrix}$$

$$\nabla f(-2,2) \cdot \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} = -6 + \cancel{8}^{\,2} \cdot \frac{3}{\cancel{4}}$$

$$= -6 + 6 = 0$$

<u>So</u>:

$\nabla f(-2,2) \perp \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}$ (90° angle), i.e. the tangent line of the level curve at $(-2,2)$.



Directional derivative

Def (Directional derivative): Let $f(x,y)$ be a function, $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ a 2-vector. Then,

$$f'_{\vec{a}} := \vec{a} \cdot \nabla f$$

"The directional derivative of f wrt. $\vec{a}$"

dot product

Ex: $f(x,y)$ as before, $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$f'_{\vec{a}} = \vec{a} \cdot \nabla f = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2x-2 \\ 2y+4 \end{bmatrix}$$

$$= 2(2x-2) + 1(2y+4)$$

$$= \dots = \underline{4x+2y}$$

A number

Can insert a point, say $(1,1)$:

$$f'_{\vec{a}}(1,1) = 4 \cdot 1 + 2 \cdot 1 = \underline{\underline{6}}$$

More about the second derivative test and optimization

NOTE: If (x^*, y^*) is a stationary point with

$$H(f)(x^*, y^*) = \begin{bmatrix} \overset{f''_{xx}}{A} & \overset{f''_{xy}}{B} \\ \underset{f''_{yx}}{B} & \underset{f''_{yy}}{C} \end{bmatrix}$$

Then,

① If $\underbrace{AC - B^2}_{\text{det. of Hessian: Local max. or local min.}} > 0$, then $AC > B^2 \geq 0$

since square

$\Rightarrow AC > 0$. But then,

i) $A, C > 0$, so $\overbrace{A + C}^{\text{trace}} > 0$ → Local min. from 2nd derivative test

$+ \cdot + = +$

ii) $A, C < 0$, so $A + C < 0$ → Local max. from 2nd. derivative test

$- \cdot - = +$

Hence, the possible cases are:

i) $A, C > 0$; local min.

ii) $A, C < 0$; local max.

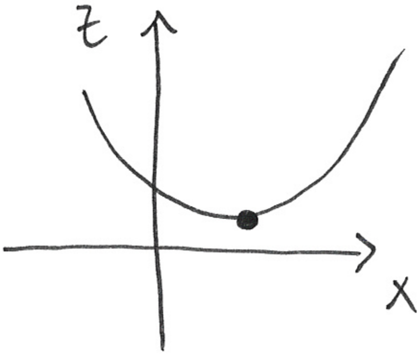
f''_{xx} $\qquad$ f''_{yy}

Recall: Second order derivative ≥ 0 ; Convex

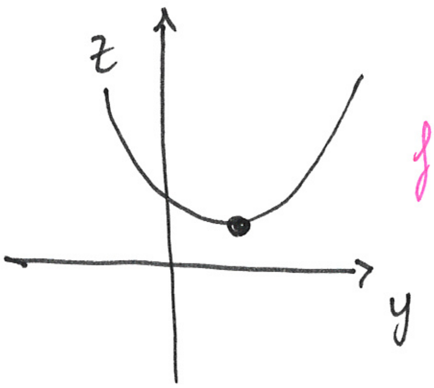
" ≤ 0 ; Concave

Graphically: Cuts of the graph $z = f(x, y)$:

i) $A, C > 0$: Locally convex cuts



$f''_{xx} > 0$

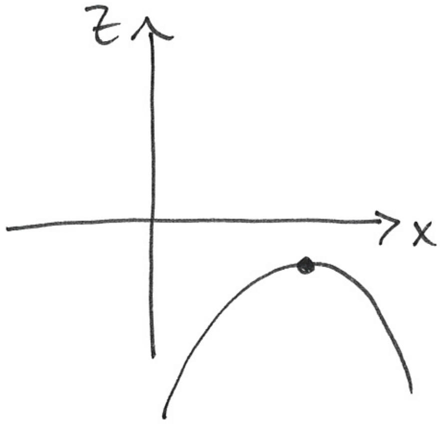


$f''_{yy} > 0$

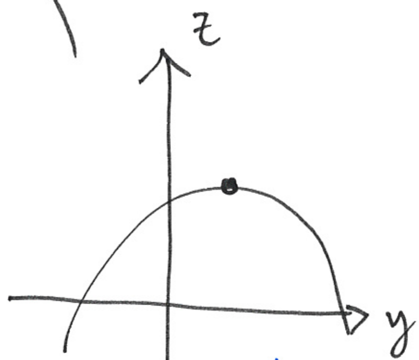
Local min.

(locally convex)

ii) $A, C < 0$: Locally concave cuts



$f''_{xx} < 0$



$f''_{yy} < 0$

Local max

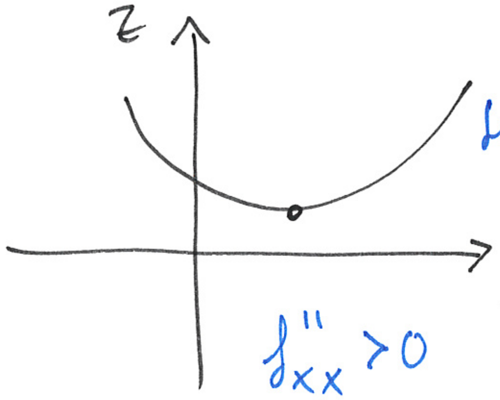
(locally concave)

② If $\underbrace{AC - B^2}_{\det H} < 0$, a typical case is

$$\underbrace{A}_{f''_{xx}} > 0, \quad \underbrace{C}_{f''_{yy}} < 0$$

$f''_{xx} > 0 \qquad f''_{yy} < 0$

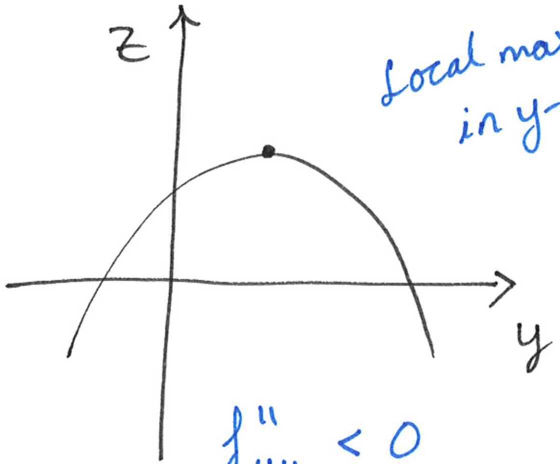
Saddle point



Local min in x-dir.

$f''_{xx} > 0$

(loc. convex)

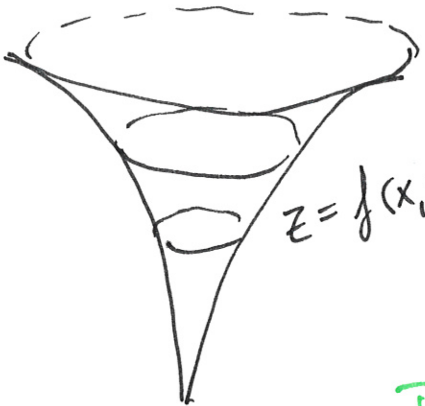


Local max in y-dir.

$f''_{yy} < 0$

(loc. concave)

Ex: $f(x,y) = \sqrt{x^2+y^2}$, $D_f = \mathbb{R}^2$ → No boundary points



$z = f(x,y)$

Stationary pts?

$$\frac{x}{\sqrt{x^2+y^2}} = f'_x = 0 \implies x = 0$$

$$\frac{y}{\sqrt{x^2+y^2}} = f'_y = 0 \implies y = 0$$

Is $(0,0)$ a stationary point? NO! Because $f'_x(0,0)$ and $f'_y(0,0)$ are not defined → divide by 0

⇒ No stationary points for f. But $(0,0)$ is a critical point, and hence a candidate point.

$$f(0,0) = \sqrt{0^2+0^2} = 0,$$

which is the smallest value f can attain.

Hence, $(0,0)$ is a global minimum for f.

minimum point