EBA1180, lecture 5, 6 Sept. 2023, Runar Ile

Plan
1. Regular cash flows
2. Infinite series and limit values
3. Euler's number and continuous compounding

1. Regular cash flows

A fixed amount is paid every period/term.

Ex Annuity loan (tot. pres. val. = what you can borrow)

Ex Saving with fixed amount each period.
Future value = the balance, what you have saved

- both give geometric series.

Ex (Term paper 2019a, probl. 6a)
Kåre considers a mortgage with
- monthly payments running for 25 year
- first payment 5 years from now
- the interest is 6%
- Kåre reckons he can pay 15000 each month

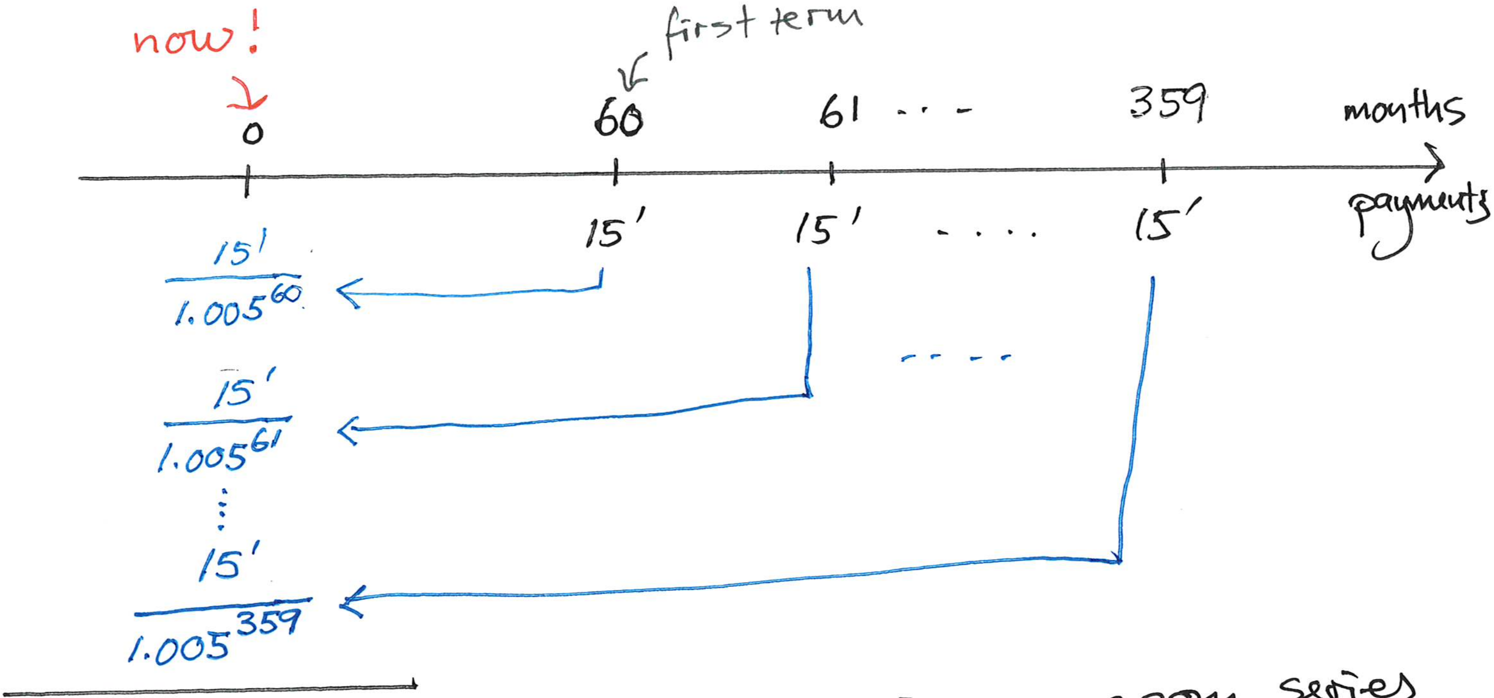
i) Determine the geom. series which gives the tot. pres. value of the cash flow

ii) Calculate how much Kåre can borrow.

Solution (both i & ii) Kåre can borrow the tot. pres. val. of the cash flow

The period rate is $\frac{6\%}{12} = 0.5\%$

Number of periods: $12 \cdot 25 = 300$



The sum (the tot. pres. val.) is a geom. series with $a_1 = \frac{15'}{1.005^{359}}$, $n = 300$, $k = 1.005$

The tot. pres. value (what Kåre can borrow) is

$$\frac{15000}{1.005^{359}} \cdot \frac{1.005^{300} - 1}{0.005} = \underline{\underline{1734620.76}}$$

2. Infinite series and limit values

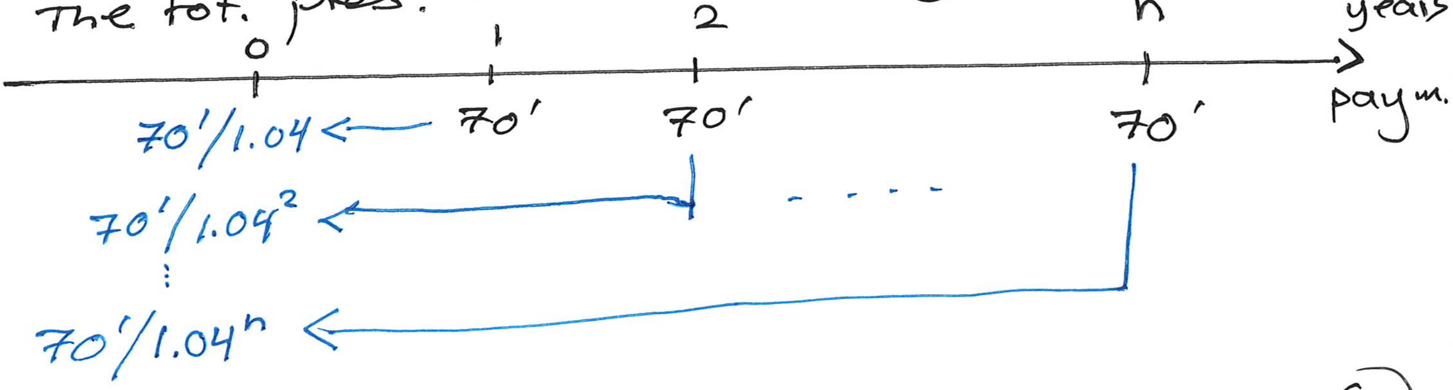
EX The annuity : 70 000

interest : 4 %

First payment : one year from now

Number of years : n

The tot. pres. value (what you can borrow)



The sum is a geom. series with

$$a_1 = \frac{70'}{1.04^n}, \quad n \text{ terms}, \quad k = 1.04$$

The sum is then: $\dfrac{70'}{1.04^n} \cdot \dfrac{1.04^n - 1}{0.04} = \dfrac{70' \cdot (1.04^n - 1)}{0.04 \cdot 1.04^n}$

$$= \frac{70' \cdot (1.04^n - 1) : 1.04^n}{0.04 \cdot 1.04^n \quad : 1.04^n} = \frac{70' \cdot \left(1 - \frac{1}{1.04^n}\right)}{0.04}$$

approaches 0 when $n \to \infty$

"n goes to infinity"

= "n becomes bigger and bigger, without bounds"

So the tot. pres. value is approaching

$$\frac{70'}{0.04} \quad \text{when } n \to \infty$$

$$= \underline{\underline{1\,750\,000}}$$

Start: 11.10

Conclusion If you pay the bank 70 000 each year, starting next year, and the interest is 4 %, and you pay forever, then you can borrow 1.75 mill.

3. Euler's number and continuous compounding

<u>Ex</u> You deposit 1000 into an account with 12% nominal interest

compounding	balance after 1 year
Annual	$1000 \cdot 1.12 = 1120.00$
Half year	$1000 \cdot 1.06^2 = 1123.60$
Quaterly	$1000 \cdot 1.03^4 = 1125.51$
Monthly	$1000 \cdot 1.01^{12} = 1126.83$
Daily	$1000 \cdot \left(1 + \frac{12\%}{365}\right)^{365} = 1127.47$

Pattern (n periods) $1000 \cdot \left(1 + \frac{0.12}{n}\right)^n$

!!

Euler's number: $e = 2.718281\ldots$

Calculator: 1 $\boxed{e^x}$

Calculate: $1000 \cdot e^{0.12} = 1127.50$

1000 $\boxed{\times}$ 0.12 $\boxed{e^x}$ $\boxed{=}$

Euler's number e is defined as the limit of $\left(1 + \frac{1}{n}\right)^n$ when $n \to \infty$

Write $\left(1 + \frac{1}{n}\right)^n \xrightarrow[n \to \infty]{} e$

<u>Ex</u>: $\left(1 + \frac{1}{1000}\right)^{1000} = 2.71692\ldots$ $\left(1 + \frac{1}{1\,\text{mill}}\right)^{1\,\text{mill}} = 2.718280\ldots$

Back to the example with 12%

$$\left(1+\frac{0.12}{n}\right)^n = \left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^n$$

$$= \Bigg[\underbrace{\left(1+\frac{1}{\left(\frac{n}{0.12}\right)}\right)^{\frac{n}{0.12}}}_{\text{approaches } e \text{ as } n \to \infty}\Bigg]^{0.12} \xrightarrow[n\to\infty]{} e^{0.12}$$

So $1000 \cdot \left(1+\frac{0.12}{n}\right)^n \longrightarrow 1000 \cdot e^{0.12}$

After 1 year with 12% nominal interest and <u>continuous compounding</u> the deposit of 1000 has increased to

$$1000 \cdot e^{0.12} = 1127.50$$

(the growth factor for 1 year with continuous compounding.

Annual growth factor : $e^{0.12} = 1.12750$

The <u>effective</u> interest : $e^{0.12} - 1 = 0.12750$

$= 12.75\%$

After 2 years of continuous compounding :

$$1000 \cdot e^{0.12} \cdot e^{0.12} = 1000 \cdot e^{0.12+0.12}$$

$$= 1000 \cdot e^{0.12 \cdot 2} = 1000 \cdot e^{0.24} = \underline{\underline{1271.25}}$$