Exercise session problems

Problem 1.
Determine the tangent line of the level curve $f(x,y) = c$ in $(x,y) = (1,1)$:

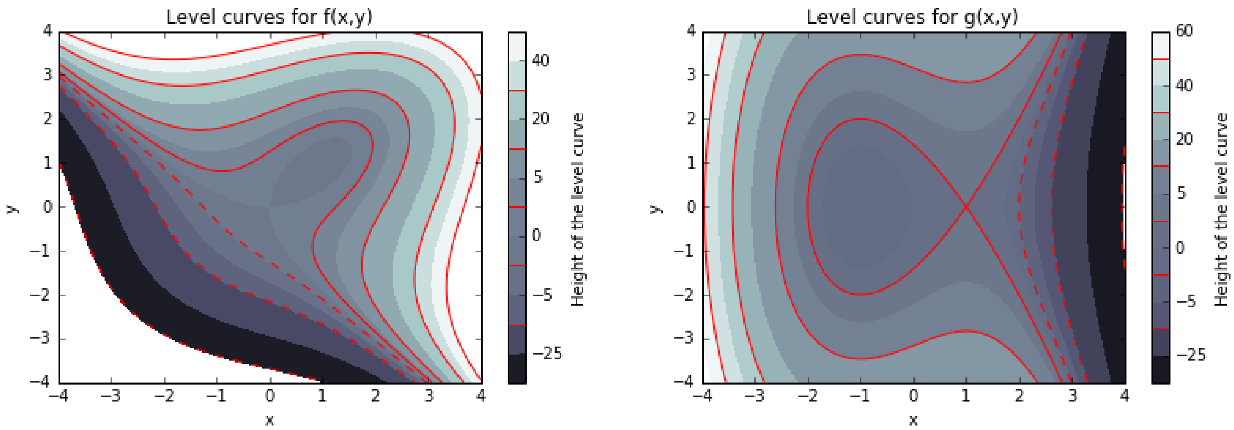
a) $f(x,y) = 2x + 3y,\ c = 5$
b) $f(x,y) = x^2 + y^2,\ c = 2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2,\ c = 7$
d) $f(x,y) = x^2 - 2x + 4y^2,\ c = 3$
e) $f(x,y) = x^3 - 3xy + y^3,\ c = -1$
f) $f(x,y) = y^2 - x^3 + 3x,\ c = 3$
g) $f(x,y) = x^2y^2 - x^2 - y^2 + 3,\ c = 2$
h) $f(x,y) = \sqrt{x^2 + y^2},\ c = \sqrt{2}$

Problem 2.
Consider the function $f(x,y) = x^2 - 2x + 4y^2$.

a) Show that the level curve $f(x,y) = c$ is an ellipse when $c > -1$, and determine the center (x_0,y_0) of the ellipse and its half axis a and b. Use this to sketch the level curves for $c = 0, 1, 2, 3$ in the same coordinate system.
b) Find the tangent lines of the level curve through $(x,y) = (1,1)$ and through $(x,y) = (2,1/2)$, and draw the tangents.
c) Find $\nabla f(1,1)$ and $\nabla f(2,1/2)$, and draw these. What happens to the function values along the gradient?
d) Does it look like the function f has a minimum- or a maximum value? Explain why/why not.

Problem 3.
Consider the level curve $f(x,y) = c$ of the function $f(x,y) = x^2 + 4x + y^2 - 2y$. What kind of curve is this? Describe the gradient of f in a point on the level curve geometrically.

Problem 4.
Level curves for two functions f and g in the area $-4 \le x,y \le 4$ are shown in the figures below.



a) Find any local minimum points, maximum points and saddle points in the figure.
b) The functions f and g are two of the functions from Problem 1 (see also Problems 8-10 from Exercise sheet 39). Which ones?

Problem 5.

Find the gradient $\nabla f(1,1)$ of f in the point $(1,1)$, and use this to find the directional derivative $f'_{\mathbf{a}}(1,1)$ of $f(x,y)$ in the point $(1,1)$ along the vector $\mathbf{a} = (a_1,a_2)$:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = x^2 + y^2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2$
d) $f(x,y) = x^2 - 2x + 4y^2$
e) $f(x,y) = x^3 - 3xy + y^3$
f) $f(x,y) = y^2 - x^3 + 3x$
g) $f(x,y) = \sqrt{x^2 + y^2}$

Problem 6.

Show that the gradient $\nabla f(a,b)$ is orthogonal to the tangent line of the level curve $f(x,y) = c$ in the point (a,b), and that f grows if we move a small step along the gradient.

Problem 7.

Find the global maximum- and minimum points, if they exist:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = x^2 + y^2$
c) $f(x,y) = 4x^2 - 6xy + 9y^2$
d) $f(x,y) = x^2 - 2x + 4y^2$
e) $f(x,y) = x^3 - 3xy + y^3$
f) $f(x,y) = y^2 - x^3 + 3x$
g) $f(x,y) = x^2y^2 - x^2 - y^2 + 3$
h) $f(x,y) = \sqrt{x^2 + y^2}$

Optional: Exercises from the Norwegian textbook

Textbook [E]:	Eriksen, *Matematikk for økonomi og finans*
Exercise book [O]:	Eriksen, *Matematikk for økonomi og finans - Oppgaver og Løsningsforslag*
Exercises:	[E] 7.4.3 - 7.4.4, 7.5.1 - 7.5.5
Solution manual:	See [O] Ch. 7.4 - 7.5

Answers to the Exercise session problems

Problem 1.

a) $y = -2x/3 + 5/3$
b) $y = -x + 2$
c) $y = -x/6 + 7/6$
d) $y = 1$
e) No tangent line.
f) $y = 1$
g) No tangent line.
h) $y = -x + 2$

Problem 2.

a) Ellipse with center in $(1,0)$ with half axis $a = \sqrt{c+1}$ and $b = \sqrt{c+1}/2$.

b) The tangent lines are given by the equation $y = 1$ and $y = -x/2 + 3/2$.

c) $\nabla f(1,1) = \begin{pmatrix} 0 & 8 \end{pmatrix}^T$, and $\nabla f(2,1/2) = \begin{pmatrix} 2 & 4 \end{pmatrix}^T$, and the function values increase when we move along the gradient.

d) No maximum value (the half axis gets bigger the bigger c is). Minimum value $f(1,0) = -1$.

Problem 3.

The curve is a circle with center in $(-2{,}1)$ and radius $\sqrt{c+5}$. The gradient points away from the center of the circle.

Problem 4.

a) f has local min. in (1,1) and saddle point in (0,0), and g has local min. in $(-1{,}0)$ and saddle point in (1,0)

b) f is the function in e) and g is the function in f)

Problem 5.

a) $\nabla f(1{,}1) = \begin{pmatrix}2 & 3\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 2a_1 + 3a_2$

b) $\nabla f(1{,}1) = \begin{pmatrix}2 & 2\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 2a_1 + 2a_2$

c) $\nabla f(1{,}1) = \begin{pmatrix}2 & 12\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 2a_1 + 12a_2$

d) $\nabla f(1{,}1) = \begin{pmatrix}0 & 8\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 8a_2$

e) $\nabla f(1{,}1) = \begin{pmatrix}0 & 0\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 0$

f) $\nabla f(1{,}1) = \begin{pmatrix}0 & 2\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = 2a_2$

g) $\nabla f(1{,}1) = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}^T$, $f'_{\mathbf{a}}(1{,}1) = (a_1 + a_2)/\sqrt{2}$

Problem 7.

a) no global max/min.

b) (0,0) is the global min.

c) (0,0) is the global min.

d) (1,0) is the global min.

e) no global max/min.

f) no global max/min.

g) no global max/min.

h) (0,0) is the global min.