EBA 1180, lecture 13, 8 Oct. 2024, Runar Ile

Plan 1. Rational functions and asymptotes
2. Hyperbolas

1. Rational functions and asymptotes

Rational function $f(x) = \frac{p(x)}{q(x)}$ polynomials

Ex $f(x) = \frac{2x+1}{x^2+3}$ - would like to see what happens when x is big.

highest degree term

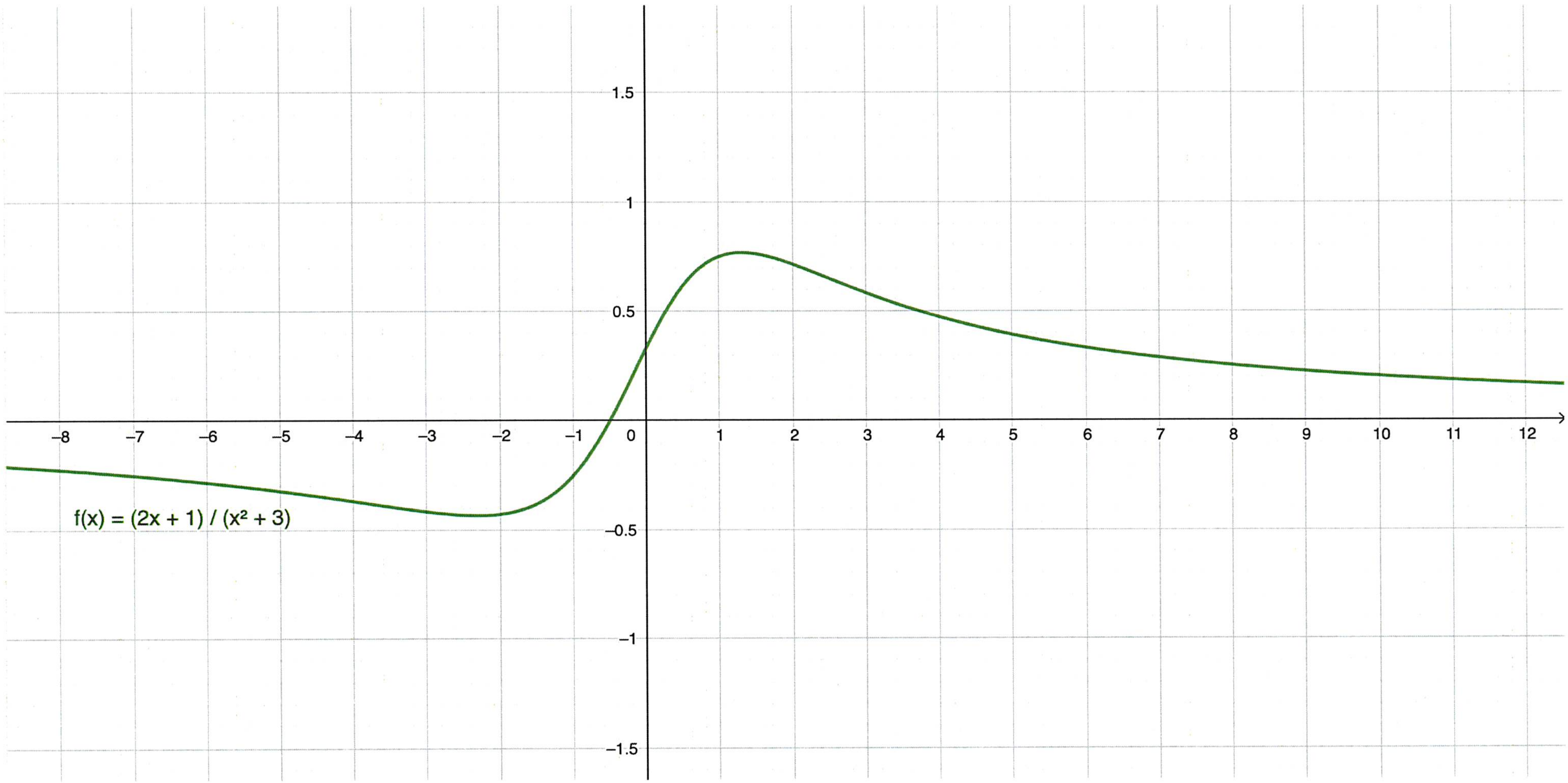
- divide by x^2 both in the numerator and in the denominator

$$\frac{\left(\frac{2x+1}{x^2}\right)}{\left(\frac{x^2+3}{x^2}\right)} = \frac{\frac{2x}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}} = \frac{\frac{2}{x}+\frac{1}{x^2}}{1+\frac{3}{x^2}} \xrightarrow[x\to\pm\infty]{} \frac{0}{1} = 0$$

$$f(1000) = \frac{\frac{2}{1000}+\frac{1}{1000^2}}{1+\frac{3}{1000^2}} = 0.00200099\ldots$$

This means that the line $\overbrace{y=0 \ (x \text{ is free})}^{\text{the } x\text{-axis}}$ is a horizontal asymptote for $f(x)$.

The graph of $f(x)$ is approaching the x-axis (the horizontal asymptote) when x becomes big. (pos/neg.)



f(x) = (2x + 1) / (x² + 3)
1.5
1
0.5
0
−0.5
−1
−1.5
−8
−7
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6
7
8
9
10
11
12

Ex $f(x) = \dfrac{2x+1}{(x-1)(x-5)}$ $\qquad (x \neq 1, x \neq 5)$

What happens when x is approaching 1 or 5?

If $x \longrightarrow 1^-$ "x is approaching 1 from below"

$x = 0.99$, $x = 0.999$, $x = 0.9999$, ...

then

$$\left.\begin{array}{l} x-1 \longrightarrow 0^- \\ x-5 \longrightarrow -4^- \\ 2x+1 \longrightarrow 3^- \end{array}\right\} \text{implies } f(x) = \frac{\overbrace{2x+1}^{3^-}}{\underbrace{(x-1)}_{0^-}\underbrace{(x-5)}_{-4^-}} \xrightarrow[x \to 1^-]{} +\infty$$

If $x \longrightarrow 1^+$ "x is approaching 1 from above"

$x = 1.1$, $x = 1.01$, $x = 1.001$, ...

$$\text{then } \left.\begin{array}{l} x-1 \longrightarrow 0^+ \\ x-5 \longrightarrow -4^+ \\ 2x+1 \longrightarrow 3^+ \end{array}\right\} \text{implies } f(x) = \frac{\overbrace{2x+1}^{3^+}}{\underbrace{(x-1)}_{0^+}\underbrace{(x-5)}_{-4^+}} \xrightarrow[x \to 1^+]{} -\infty$$

Conclusion The line $x = 1$ (y is free) is a vertical asymptote for $f(x)$.

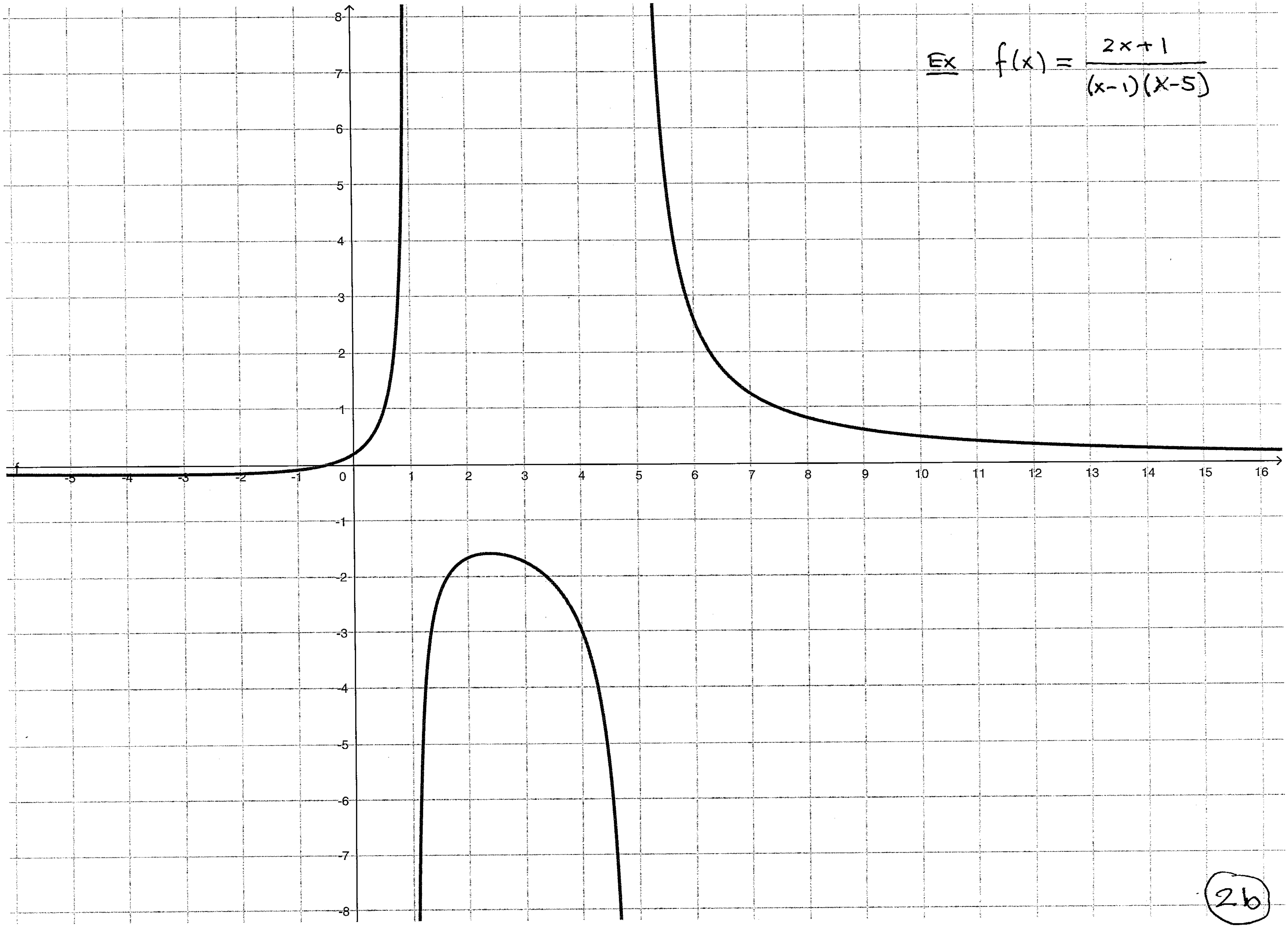
The graph of $f(x)$ is approaching the vertical line $x = 1$ when $x \to 1$.

Note The line $x = 5$ (y free) is another vertical asymptote for $f(x)$:

$$f(x) \xrightarrow[x \to 5^-]{} -\infty, \qquad f(x) \xrightarrow[x \to 5^+]{} +\infty$$

In addition: $f(x)$ also has a horizontal asymptote $y = 0$ (so the x-axis)

Ex $f(x) = \dfrac{2x+1}{(x-1)(x-5)}$



2b

Non-vertical (oblique) asymptotes

Ex $f(x) = x-5 + \frac{2}{x-4}$ has a vertical asymptote $x=4$.

But also an oblique asymptote:

Put $g(x) = x-5$.

Then the graph of $f(x)$ is approaching the graph of $g(x)$ when $x \longrightarrow \pm\infty$

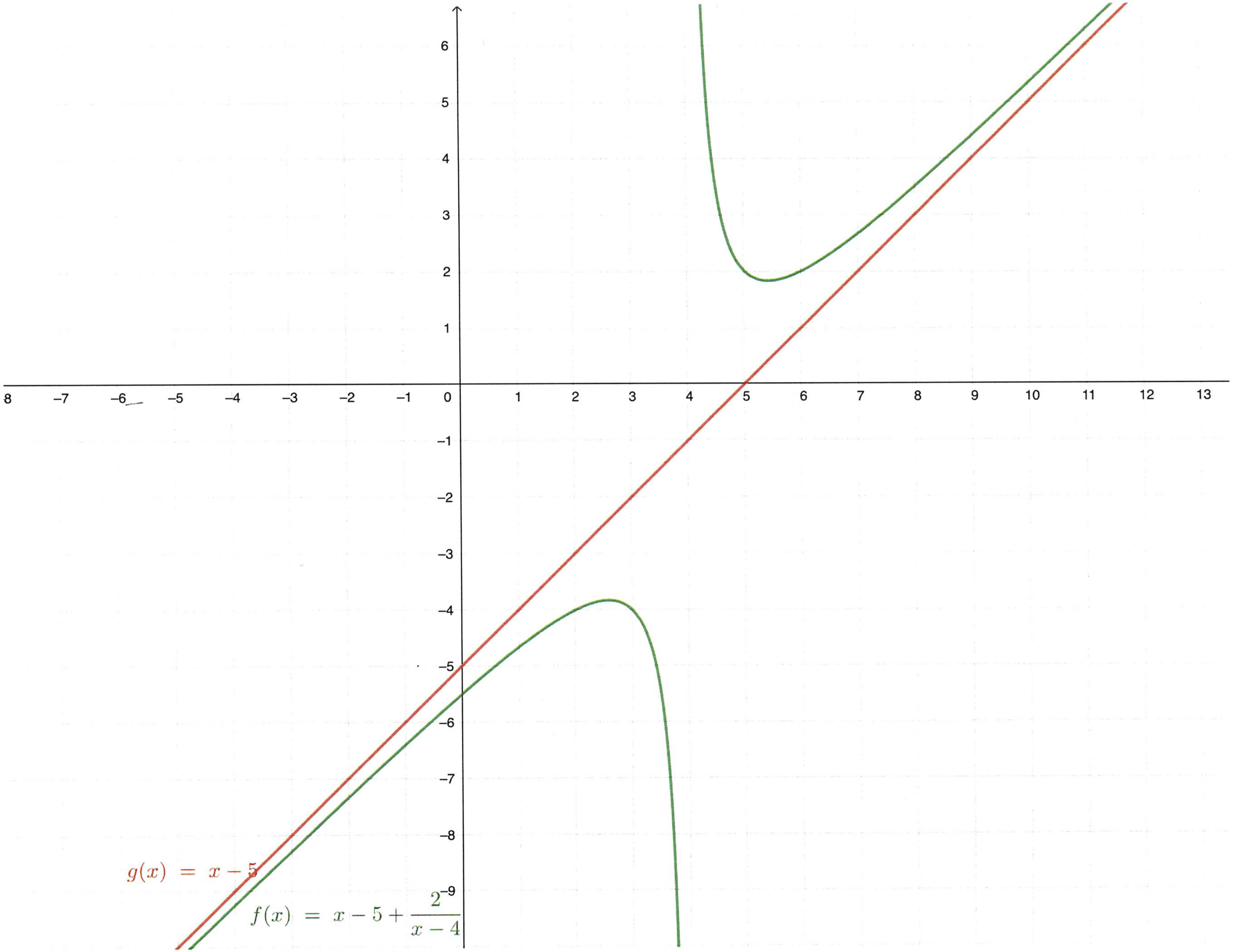
because $f(x)-g(x) = \frac{2}{x-4} \xrightarrow[x\to\pm\infty]{} 0$

Note that $f(x) = \frac{(x-5)\cdot(x-4)+2}{x-4} = \frac{x^2-9x+22}{x-4}$

- have to do polynomial division to find the better expression $\underbrace{x-5}_{g(x)} + \frac{2}{x-4}$

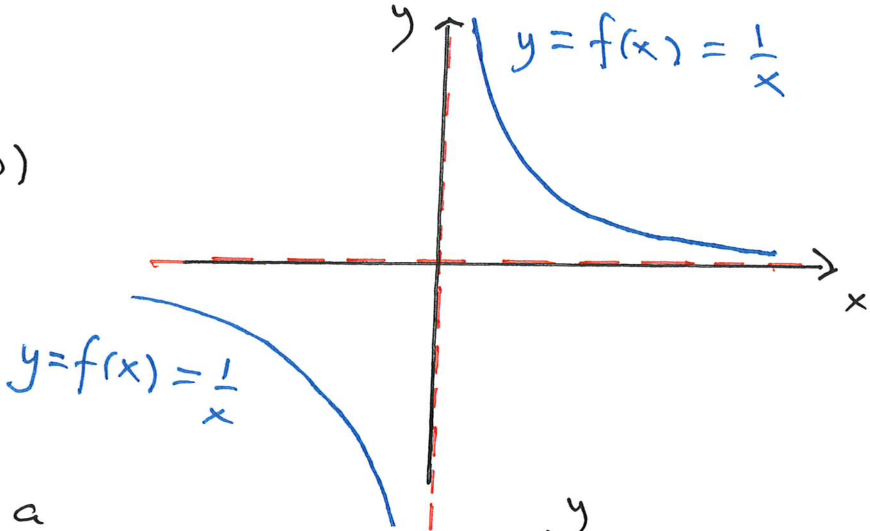
The graph of $g(x)$ is a non-vertical asymptote for $f(x)$.

Start: 11.07



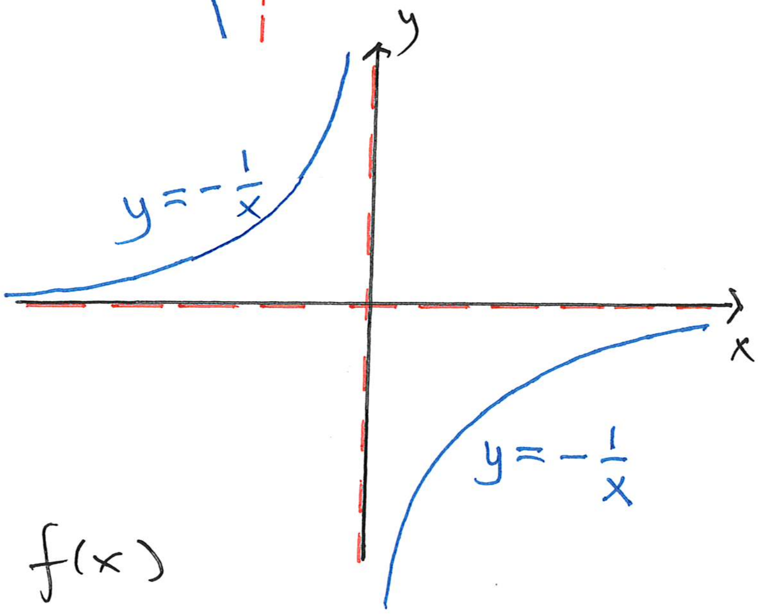
3b

2. Hyperbolas



Ex $f(x) = \frac{1}{x} \quad (x \neq 0)$

The line $y = 0$ is a horizontal asymptote.
The line $x = 0$ is a vertical asymptote.



Ex $f(x) = -\frac{1}{x} \quad (x \neq 0)$

– same asymptotes.

Definition A function $f(x)$ is a hyperbola function if it can be written as

$$f(x) = c + \frac{a}{x-b} \qquad (a \neq 0)$$

Ex $f(x) = \frac{3x-5}{x-2}$ is a hyperbola function because polynomial division gives

$$(3x-5) : (x-2) = 3 + \frac{1}{x-2} \quad \text{so} \quad a = 1,\ b = 2,\ c = 3$$

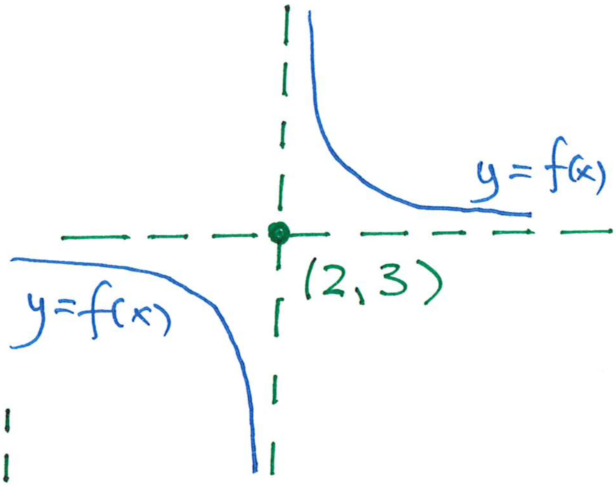
$$\begin{array}{l} -(3x-6) \\ \hline \quad 1 \end{array} \qquad \cdot (x-2)$$

$$\text{So } f(x) = 3 + \frac{1}{x-2}$$

We have $3 + \frac{1}{x-2} \xrightarrow[x \to 2^-]{} -\infty$ and $3 + \frac{1}{x-2} \xrightarrow[x \to 2^+]{} +\infty$

Also note that $3 + \frac{1}{x-2} \xrightarrow[x \to \pm\infty]{} 3^{\pm}$

Conclusion: The line $x = 2$ is a vertical asymptote and the line $y = 3$ is a horizontal —||—



$$f(x) = 3 + \frac{1}{x-2}$$

$$f(1) = 3 + \frac{1}{1-2} = 2$$

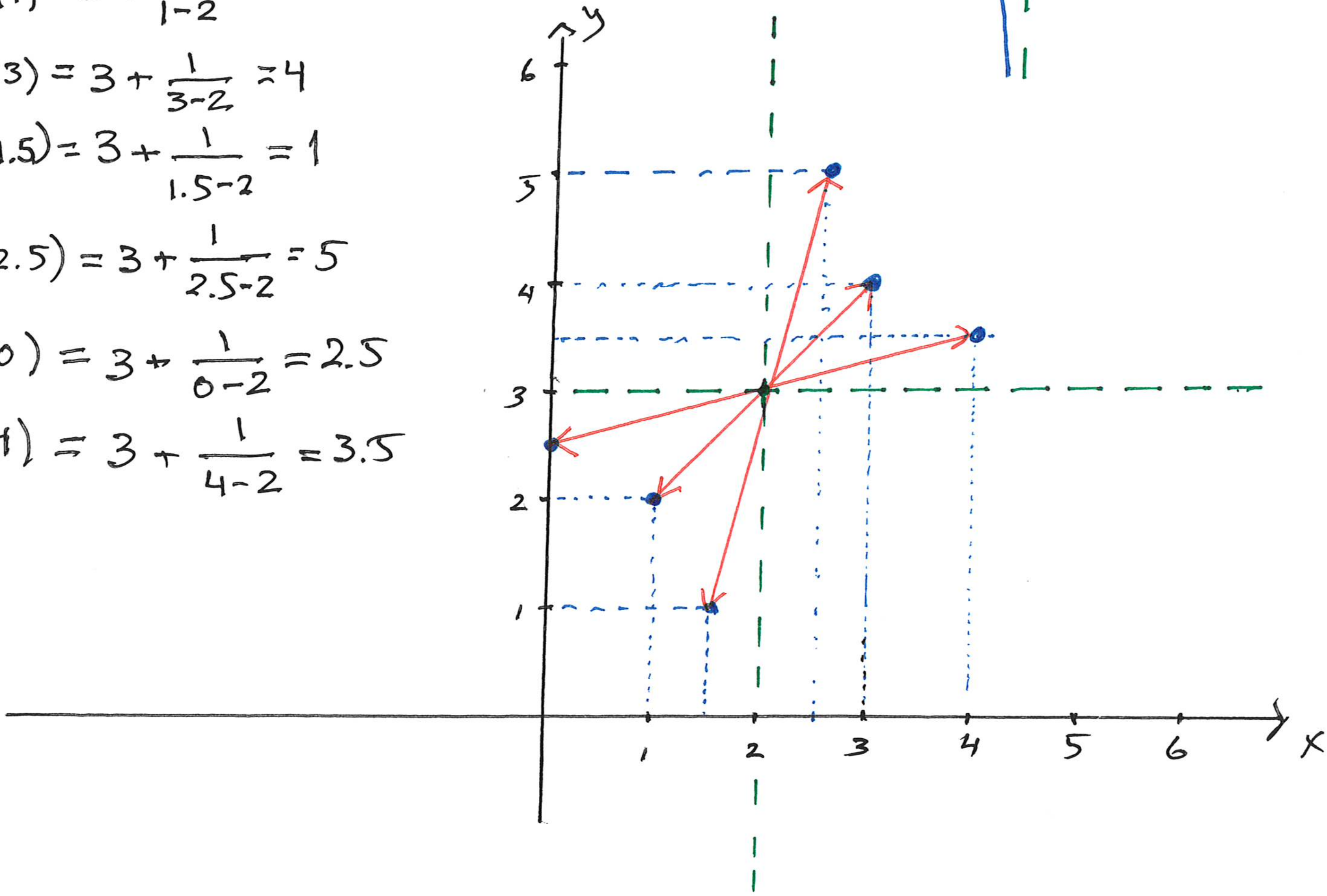
$$f(3) = 3 + \frac{1}{3-2} = 4$$

$$f(1.5) = 3 + \frac{1}{1.5-2} = 1$$

$$f(2.5) = 3 + \frac{1}{2.5-2} = 5$$

$$f(0) = 3 + \frac{1}{0-2} = 2.5$$

$$f(4) = 3 + \frac{1}{4-2} = 3.5$$



The graph of a hyperbole function is symmetric through the intersection point of the asymptotes!

2019s Multiple Choice

Problem 5

We have the hyperbola function $f(x) = \dfrac{4x-38}{x-10}$. Which of the graphs in figure 1 is the graph of $f(x)$?

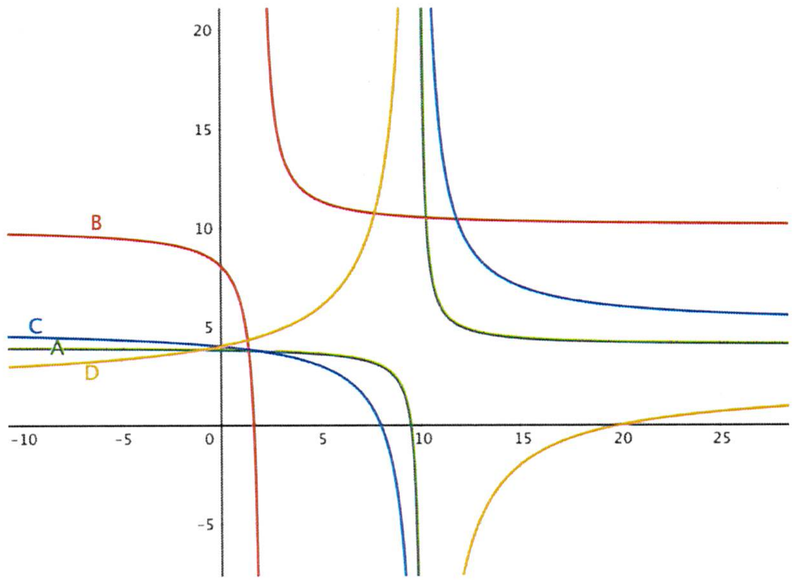


Figure 1: Graphs A-D

(A) $f(x)$ has the graph A (green)
(B) $f(x)$ has the graph B (red)
(C) $f(x)$ has the graph C (blue)
(D) $f(x)$ has the graph D (yellow)
(E) I choose not to answer this problem.

5b

2019 a Term Paper

Find the expression for the hyperbola function.

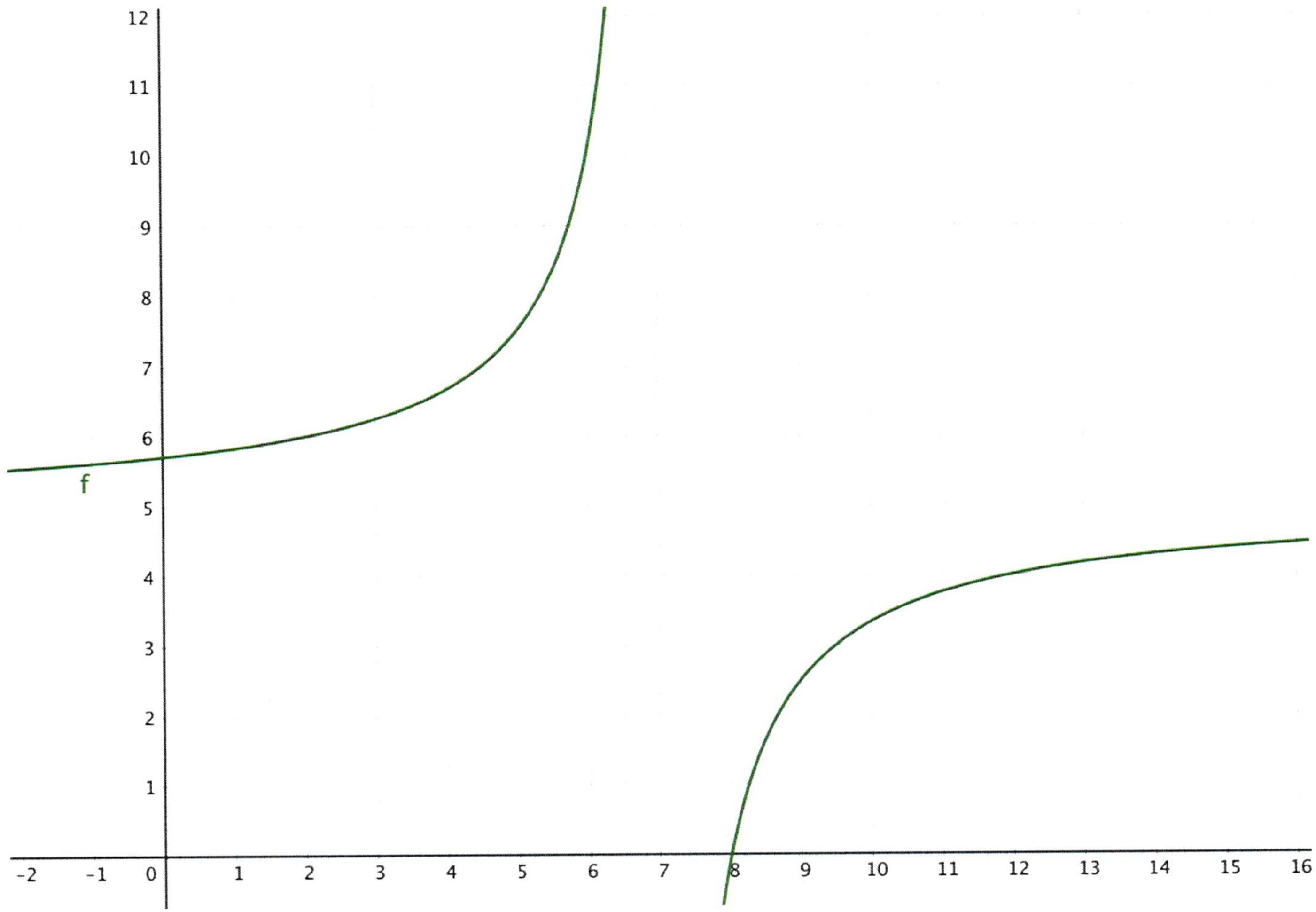


Figure 2: Hyperbola

5c