EBA 1180, lecture 21, 5 Nov. 2024, Runar Ile

Plan : 1. Repetition (problems from last week)
1d) implicit differentiation
2) implicitly defined curves
6b) concave/convex functions
8b) convex optimization
2. l'Hopitals rule

1. Repetition

1d) Implicit differentiation.

$$x^3 - 3xy + y^2 = 0 \qquad (*)$$

We find an expression for y' in y and x by differentiating each side of $(*)$ with respect to x, and then solve for y'.

Partial calculations

$$(xy)'_x \overset{\text{prod. rule}}{=} (x)'_x \cdot y + x \cdot y'_x$$

$$= 1 \cdot y + x \cdot y' = y + xy'$$

$$(y^2)'_x \overset{\text{chain rule}}{=} 2y \cdot y'$$

Then $(*)$ gives

$$3x^2 - 3(y + xy') + 2yy' = 0$$

Solve this eq. for y':

$$3x^2 - 3y - 3x\underline{y'} + 2y\underline{y'} = 0$$

$$(2y - 3x)y' = 3(y - x^2)$$

$$y' = \frac{3(y-x^2)}{(2y-3x)} \qquad (**)$$

EBA1180 Mathematics for Data Science
autumn 2024
Exercises

I came to the position that mathematical analysis is not one of the many ways of doing economic theory: it is the only way.

R. Lucas

Lecture 19 – 20
Sec. 7.1, 6.9, 8.6-7:
Implicit differentiation. The second order derivative, convex/concave functions.

Here are recommended exercises from the textbook [SHSC].

Section **7.1** exercise 1, 4, 6, 7a
Section **6.9** exercise 1-4
Section **9.6** exercise 1-4, 6a
Section **8.6** exercise 1-4

Problems for the exercise session
Wednesday 30 Oct. 12–14+

Problem 1 Find an expression for y' in terms of y and x by implicit differentiation. Find all solutions for y with $x = a$ and determine the expression for the tangent function in each of these points.

a) $x^2 + 25y^2 - 50y = 0$ and $a = 4$

b) $x^{3.27}y^{1.09} = 1$ and $a = 1$

c) $x^4 - x^2 + y^4 = 0$ and $a = \frac{\sqrt{2}}{2}$

d) $x^3 - 3xy + y^2 = 0$ and $a = 2$

Problem 2 in figure 1 you see the graphs of the implicit defined curves in Problem 1. Determine the curves and the equations which belong together. Also draw the tangents in Problem 1.

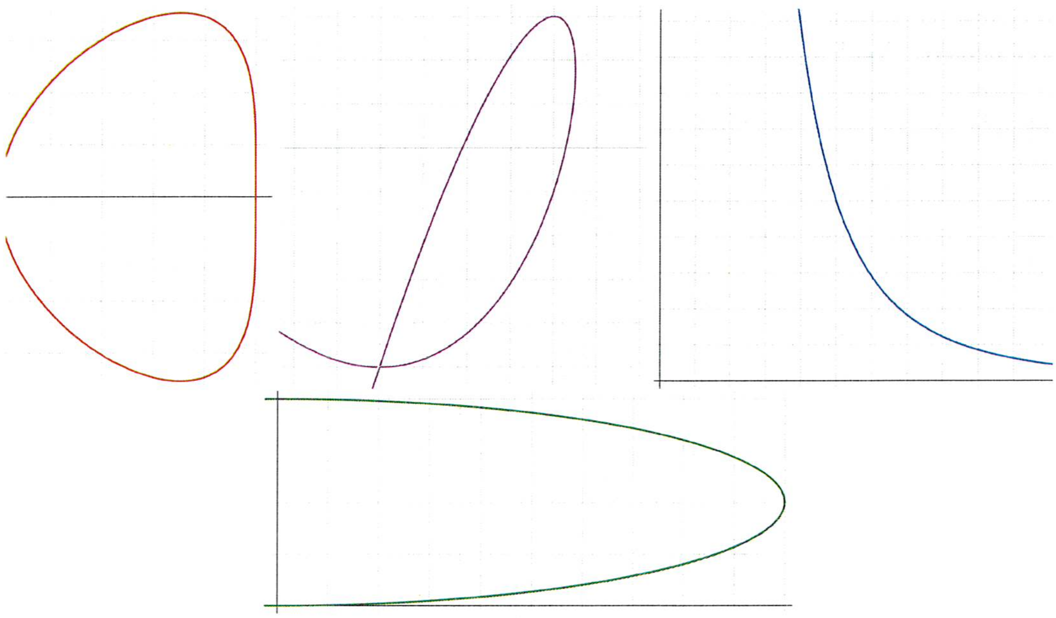


Figure 1: Four implicitly defined curves

- Assume $\underline{x=2}$, then we find the possible y-values by solving $(*)$ with $x=2$:

$$2^3 - 3\cdot 2\cdot y + y^2 = 0$$
$$y^2 - 6y = -8$$
$$(y-3)^2 = -8+9 = 1$$

so either $y-3=1$ or $y-3=-1$

that is $\underline{\underline{y=4}}$, $\underline{\underline{y=2}}$

- We use the point-slope formula to find the two tangent functions through the points $(2,4)$ and $(2,2)$

$(2,4)$: $$y' = \frac{3(4-2^2)}{2\cdot 4 - 3\cdot 2} = 0$$

so the tangentfunction is constant: $\underline{\underline{h_1(x) = 4}}$

$(2,2)$: $$y' = \frac{3(2-2^2)}{2\cdot 2 - 3\cdot 2} = \frac{3\cdot(-2)}{-2} = 3$$

So the point-slope formula gives

$$h_2(x) - 2 = 3(x-2)$$

$$\underline{\underline{h_2(x) = 3x-4}}$$

2) Elimination is the strategy.

① In 1a, c and d we got <u>two</u> y-values for one x-value. So 1a, c and d cannot be the blue one.

So <u>1b has to be the blue one.</u>

② The red and the green graphs are symmetric (around horizontal lines), so their tangents are symmetric too.

In particular the slopes are only changing signs (for fixed x-values).

This is the case for 1a and c.

So <u>1d has to be the purple one</u> (in the middle)

③ In 1a we have both y-values positive

In 1c one y-value is negative.

If the thicker horizontal lines are the x-axes, then

<u>1a has to be the green</u> (bottom) one

<u>1c — " — red</u> (upper left) one

6b) $f(x) = \ln(x^2 - 2x + 2) - \frac{x}{4} + 1$

<u>Note</u> $x^2 - 2x + 2 = (x-1)^2 + 1 \geq 1$ so

$f(x)$ is defined for all values of x.

$$f'(x) = \left[\ln(x^2 - 2x + 2)\right]' - \frac{1}{4} + 0$$

Chain rule with

$u = x^2 - 2x + 2$ and $g(u) = \ln(u)$

$u'(x) = 2x - 2$ $\qquad g'(u) = \frac{1}{u}$

$$= \frac{2x - 2}{x^2 - 2x + 2} - \frac{1}{4}$$

$$f''(x) = \frac{(2x-2)'(x^2-2x+2) - (2x-2)\cdot(x^2-2x+2)'}{(x^2-2x+2)^2}$$

$$= \frac{2\cdot(x^2-2x+2) - \overbrace{(2x-2)\cdot(2x-2)}^{4x^2-8x+4}}{(x^2-2x+2)^2}$$

$$= \frac{2x^2 - 4x + 4 - 4x^2 + 8x - 4}{(x^2-2x+2)^2}$$

$$= \frac{-2x^2 + 4x}{(x^2-2x+2)^2} = \frac{-2x(x-2)}{\left[(x-1)^2 + 1\right]^2}$$

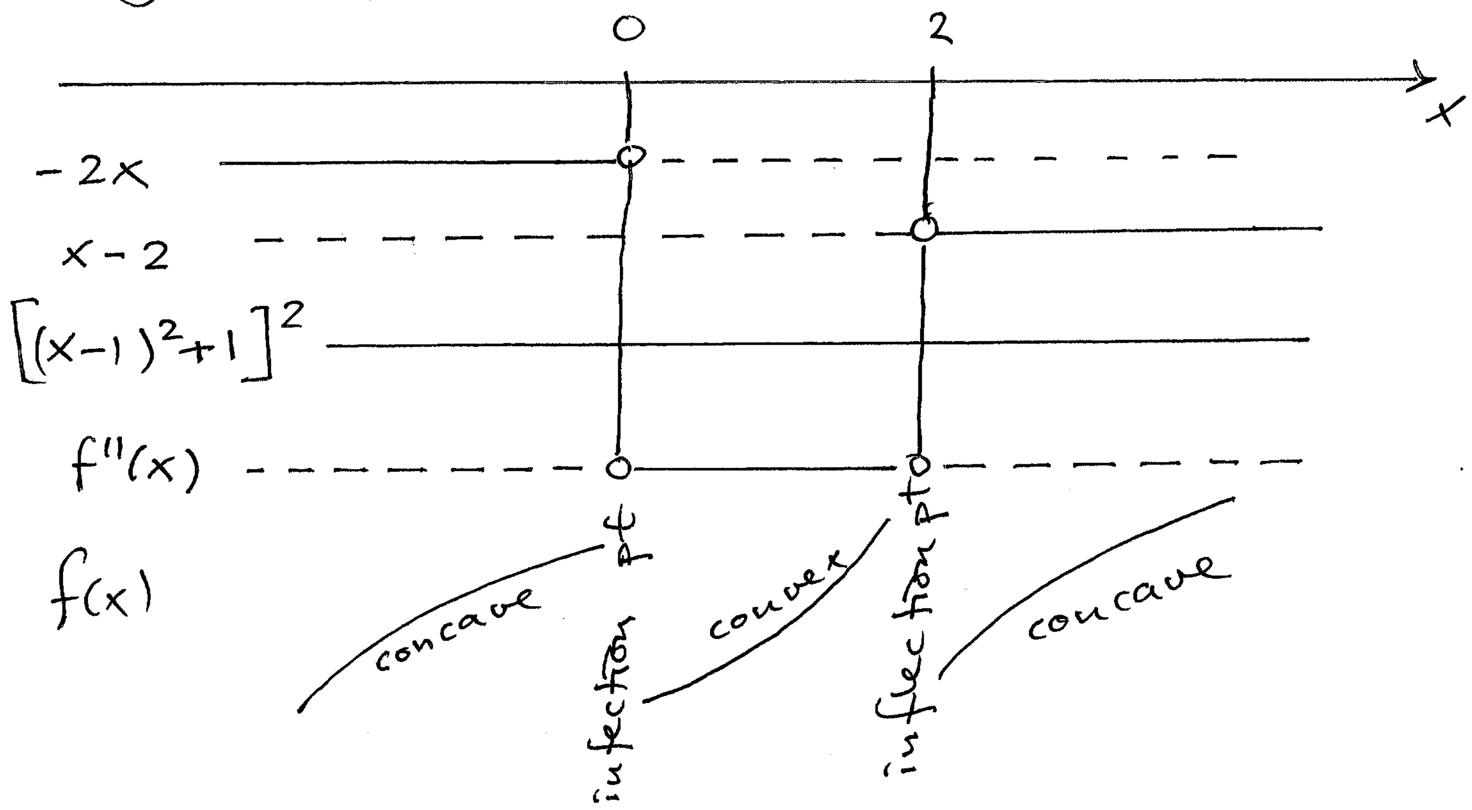
Solve eq. $f''(x) = 0$,

that is $-2x(x-2) = 0$

so $-2x = 0$ or $x - 2 = 0$

that is $\underline{\underline{x = 0}}$ or $\underline{\underline{x = 2}}$

Sign diag. for $f''(x)$.



Conclusion

$f(x)$ is concave for x in $\langle \leftarrow, 0]$

— " — convex — " — $[0, 2]$

— " — concave — " — $[2, \rightarrow\rangle$

Then the inflection points for $f(x)$

are $\underline{\underline{x = 0}}$ and $\underline{\underline{x = 2}}$

because $f''(x)$ changes sign here.

8b) • Determine the (loc.) max/min. point for f(x). (Start: 11.03)

• Explain why they give global max/min by using convexity.

• Calculate max/min. of f(x).

$$f(x) = \frac{-1}{x(x-6)} , \quad D_f = \langle 0, 6 \rangle$$

• $f'(x) \overset{\text{frac. rule}}{=} \dfrac{2x-6}{[x(x-6)]^2}$ Stationary points: Solutions of eq. $f'(x) = 0$

that is $2x-6=0$ $\quad (x(x-6) \neq 0)$

$\underline{x = 3}$ $\quad (3\cdot(3-6) \neq 0 \ - \text{ok})$

Numerator of $f'(x)$ is changing sign from $-$ to $+$ at $x=3$, and denominator is pos. so $f'(x)$ changes sign from

$-$ to $+$ at $x=3$.

and $\underline{\underline{x=3}}$ is a (loc.) min. point

• Calculate $f''(x) = \left[\dfrac{2x-6}{x^2(x-6)^2}\right]'$

$$\left[x^2\cdot(x-6)^2\right]' = 2x\cdot(x-6)^2 + x^2\cdot 2(x-6)\cdot 1$$
$$= 2x(x-6)(x-6+x)$$
$$= 2x(x-6)(2x-6)$$

$$f''(x) = \frac{2\cdot x^{\cancel{2}}\cdot(x-6)^{\cancel{2}} - (2x-6)\cdot 2\cancel{x}\,\cancel{(x-6)}\,(2x-6)}{x^{\cancel{4}\,3}\,(x-6)^{\cancel{4}\,3}}$$

$$= \frac{2x(x-6) - (2x-6)^2\cdot 2}{x^3(x-6)^3}$$

$$= \frac{2(-3x^2+18x-36)}{x^3(x-6)^3}$$

$$= \frac{-6(x^2-6x+12)}{x^3(x-6)^3} = \frac{-6\left[(x-3)^2+3\right]}{x^3(x-6)^3}$$

For $x \in \langle 0, 6\rangle$, then $x^3 > 0$ and

$(x-6)^3 < 0$

So $f''(x) = \frac{\text{neg.}}{\text{neg.}} > 0$ for $x \in D_f = \langle 0, 6\rangle$

Hence $f(x)$ is convex in the whole domain and $x = 3$ is a global minimum point.

- Minimal value of $f(x)$ is

$$f(3) = \frac{-1}{3(3-6)} = \frac{-1}{-9} = \underline{\underline{\frac{1}{9}}}$$

– no maximal value $\left(f(x) \xrightarrow[x\to 0^+]{} +\infty \right.$

$\left. f(x) \xrightarrow[x\to 6^-]{} +\infty \right)$

2. l'Hopital's rule

It is about limits of the type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

Notation $\lim_{x\to 5} f(x)$ is the number which $f(x)$ is approaching when x is approaching 5.

Ex $f(x) = \dfrac{3x-3}{\ln(x)}$. Want to find $\lim_{x\to 1} f(x)$

$$\left.\begin{array}{ll}\text{Numerator:} & 3x-3 \longrightarrow 3\cdot 1 - 3 = 0 \\ \text{Denominator:} & \ln(x) \longrightarrow \ln(1) = 0\end{array}\right\} \frac{0}{0}\text{-expr.}$$

Then we can use l'Hôpital's rule

$$\lim_{x\to 1} f(x) \overset{\text{l'Hôp}}{=} \lim_{x\to 1} \frac{(3x-3)'}{[\ln(x)]'} = \lim_{x\to 1} \frac{3}{\left(\frac{1}{x}\right)} = \frac{3}{\left(\frac{1}{1}\right)} = \underline{\underline{3}}$$

Check: $f(1.01) = \dfrac{3\cdot 1.01 - 3}{\ln(1.01)} = 3.050$

$f(0.99) = \dfrac{3\cdot 0.99 - 3}{\ln(0.99)} = 2.9850$

Note Has to be $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$!

Then differentiate numerator and denominator seperately and try to find the limit of the new fraction.

$\underline{\text{Ex}}$ $\lim\limits_{x \to 0} \dfrac{3x}{e^x - 1}$ $\left(\dfrac{0}{0} \; ; \; 3x \xrightarrow[x \to 0]{} 0 \quad e^x - 1 \xrightarrow[x \to 0]{} 1 - 1 = 0 \right)$

$$\stackrel{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{3}{e^x} = \frac{3}{1} = \underline{\underline{3}}$$

$\underline{\text{Ex}}$ $\lim\limits_{x \to \infty} \underbrace{\dfrac{x^2}{e^x}}_{\text{"}\frac{\infty}{\infty}\text{"}} \stackrel{\text{l'Hôp}}{=} \lim\limits_{x \to \infty} \underbrace{\dfrac{2x}{e^x}}_{\text{"}\frac{\infty}{\infty}\text{"}} \stackrel{\text{l'Hôp}}{=} \lim\limits_{x \to \infty} \dfrac{2}{e^x} = \underline{\underline{0}}$

<u>Meaning</u> $f(x) = \dfrac{x^2}{e^x}$ has $y = 0$ as horizontal asymptote.