Integration

EBA1180
Spring 25
Sect. 2

WARM-UP:

Ex: i) $\int 3x^5\,dx$

ii) $\int 3x^5 + 6x^{12}\,dx$

INTEGRATION RULES:

i) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$, $n \neq -1$

iii) Integral of sum is sum of integrals

iv) $\int c\,u(x)\,dx = c\int u(x)\,dx$ (c is a constant)

oo

i) $\int 3x^5\,dx = 3\int x^5\,dx$

iv)

$= 3\cdot\dfrac{x^{5+1}}{5+1} + C$

i)

$= 3\,\dfrac{x^6}{6} + C = \dfrac{1}{2}x^6 + C$

CHECK: $\left(\frac{1}{2}x^6 + C\right)' = 3x^5$ OK!

ii) $\int 3x^5 + 6x^{12}\,dx = \int 3x^5\,dx + \int 6x^{12}\,dx$

iii)

$= \dfrac{1}{2}x^6 + 6\,\dfrac{x^{13}}{13} + C$

prev. exercise + iv) and i)

$= \dfrac{1}{2}x^6 + \dfrac{6}{13}x^{13} + C$

CHECK: $\left(\frac{1}{2}x^6 + \frac{6}{13}x^{13} + C\right)' = 3x^5 + 6x^{12}$

Ex: $\int \sqrt{x}\, dx = \int x^{\frac{1}{2}} = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

"TRICK"

power rule

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \dfrac{2}{3} x^{\frac{3}{2}} + C$

$\frac{1}{2}+1 = \frac{1}{2}+\frac{2}{2} = \frac{3}{2}$

$= \dfrac{2}{3} x\sqrt{x} + C$

$x^{\frac{3}{2}} = x^{1+\frac{1}{2}} = x^1 x^{\frac{1}{2}} = x\sqrt{x}$

Ex:

$\int \dfrac{1}{x^2}\, dx = \int x^{-2}\, dx = \dfrac{x^{-2+1}}{-2+1} + C$

power rule

$= \dfrac{x^{-1}}{-1} + C$

$= -\dfrac{1}{x} + C$

Ex: $\int \dfrac{x^2-2x+3}{x}\, dx = \int \dfrac{x^2}{x} - \dfrac{2x}{x} + \dfrac{3}{x}\, dx$

$= \int x - 2 + 3 \cdot \dfrac{1}{x}\, dx$

$= \dfrac{1}{2}x^2 - 2x + 3\ln|x| + C$

Integration technique: Substitution

Integration version of chain rule

Exp.-rule:

$$\int e^u \, du = e^u + C$$

Ex: $\int e^{2x} dx = \int e^{u} dx$

$\boxed{u = 2x}$ Make e^{2x} look like e^u

PROBLEM: Not same variable ⇝ Need du

$$= \int e^u \frac{1}{u'} du = \int e^u \frac{1}{2} du$$

$u' \to 2$

$$\frac{du}{dx} = u'$$

$$du = u' dx$$

$$dx = \frac{du}{u'}$$

$\hookrightarrow u = 2x \Rightarrow u' = 2$

$$= \frac{1}{2} \int e^u du$$

$$= \frac{1}{2} e^u + C$$

exp-rule

$$= \frac{1}{2} e^{2x} + C$$

CHECK: $\left(\frac{1}{2} e^{2x} + C\right)' = \frac{1}{2} e^{2x} \cdot (2x)' + 0$

CHAIN RULE

$$= \frac{1}{\cancel{2}} e^{2x} \cdot \cancel{2} = e^{2x}$$

original integrand: OK!

FORMULA: (Substitution)

$$du = u' dx$$

$$dx = \frac{1}{u'} du$$ where u' means the derivative of $u = u(x)$ w.r.t. x

Ex: $\int x\sqrt{x^2+1}\,dx = \int \cancel{x}\sqrt{\underbrace{u}_{x^2+1}}\,\underbrace{\frac{1}{2\cancel{x}}\,du}_{dx}$

$u = x^2+1$
$du = 2x\,dx$
$dx = \frac{1}{2x}\,du$

$= \int \frac{1}{2}\sqrt{u}\,du \underset{\boxed{\text{TRICK}}}{=} \frac{1}{2}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \frac{\cancel{1}}{\cancel{2}}\,\frac{\cancel{2}}{3}\,u^{\frac{3}{2}} + C$

$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$

Q: $\int x e^{-x^2}\,dx = \int \cancel{x}\,e^{u}\left(-\frac{1}{2\cancel{x}}\right)du$

$u = -x^2$
$du = -2x\,dx$
$dx = -\frac{1}{2x}\,du$

$= -\frac{1}{2}\int e^{u}\,du$

$= -\frac{1}{2}e^{u} + C$

$= -\frac{1}{2}e^{-x^2} + C$

EXTRA:
$\int \frac{\ln x}{x}\,dx$

EXTRA EXTRA:
$\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx$

Extra:

$$\int \frac{\ln x}{x}\,dx = \int \frac{u}{\cancel{x}}\,\cancel{x}\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$dx = x\,du$

$$= \int u\,du$$

$$= \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}(\ln x)^2 + C$$

Extra extra:

$$\int \frac{e^{1-\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{\cancel{\sqrt{x}}}\,(-2\cancel{\sqrt{x}})\,du$$

$u = 1 - \sqrt{x}$

$ = 1 - x^{\frac{1}{2}}$

$du = -\frac{1}{2}x^{-\frac{1}{2}}\,dx$

$ = -\frac{1}{2\sqrt{x}}\,dx$

$dx = -2\sqrt{x}\,du$

$$= -2\int e^u\,du$$

$$= -2e^u + C$$

$$= -2e^{1-\sqrt{x}} + C$$

Ex:

$$\int e^{\sqrt{x}}\,dx = \int e^u\,2\overbrace{\sqrt{x}}^{u}\,du$$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,du$

$$= 2\int e^u \cdot u\,du$$

"TRICK"

PRODUCT $\rightsquigarrow$ Integration by parts

Corresponds to the product rule for differentiation

$$= 2(ue^u - e^u) + C$$

Integration by parts: To be explained

$$= 2\sqrt{x}\, e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

$$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$$

METHOD: Integration by parts

FORMULA: $\int u' \cdot v \, dx = uv - \int u v' dx$

Functions of x

Hence, in example: $\int \underbrace{x}_{v} \cdot \overbrace{e^x}^{u'} dx$

1) Decide which factor can most easily be anti-diff'ed: Call this u'.

Ideally: v' is simple.

2) Anti-differentiate u': Get u

Differentiate v: Get v'

Ex: $u' = e^x \Rightarrow u = e^x$

$v = x \Rightarrow v' = 1$

3) Use formula:

Ex: $\int e^x x \, dx = e^x x - \int e^x \cdot 1 \, dx$

$= x e^x - \underbrace{\int e^x dx}_{\text{EASY!}}$

$= x e^x - e^x + C$

Why does integration by parts hold?

Product rule for differentiation

$(u \cdot v)' = u' \cdot v + u \cdot v'$

Functions of x

$\Downarrow$ Integrate on both sides

$u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' dx$

So:

INTEGRATION BY PARTS FORMULA!

$$\boxed{\int u' \cdot v \, dx = u \cdot v - \int u \cdot v' dx}$$

→ Don't need to remember the formula: Remember the product rule and this derivation.

$\boxed{\text{D3 - 037}}$ ← TA session: GO THERE!