EBA1180
Spring 25
Lecture 5
(29)

Warm-up: $\displaystyle\int \frac{5}{4-9x^2}\,dx$

What's the plan to solve this integral?

Partial fractions: $4-9x^2 = (2-3x)(2+3x)$

$$5\int \frac{1}{(2-3x)(2+3x)}\,dx$$

$$\frac{1}{4-9x^2} = \frac{1}{(2-3x)(2+3x)} = \frac{A}{2-3x} + \frac{B}{2+3x}$$

$\vdots$

Get 2 linear eqns. with two unknowns:

$A = \ldots$, $B = \ldots$

Then substitution / ln-antidiff.

Definite integrals

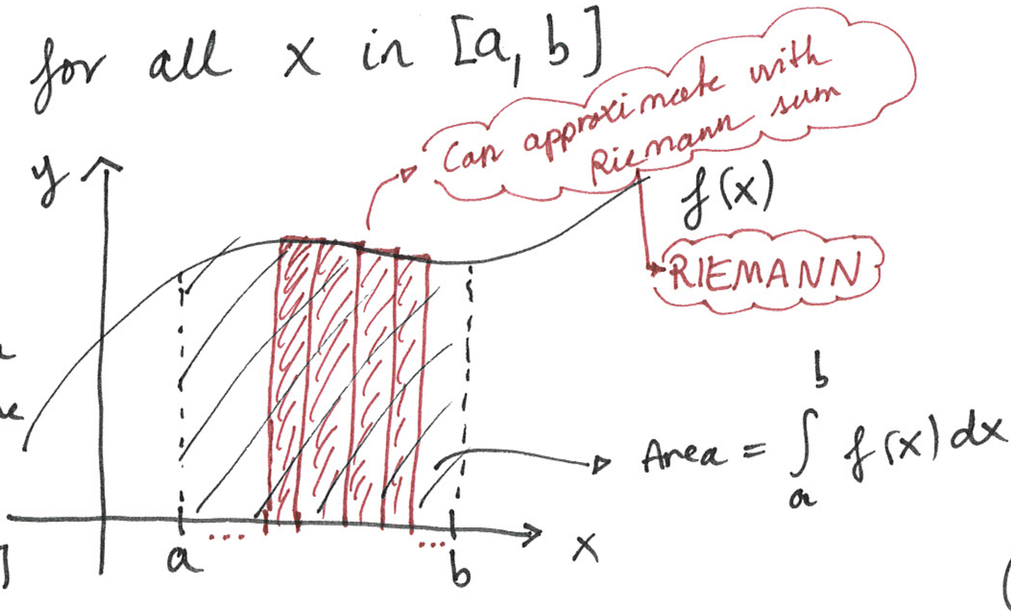
Assume: i) f is a continuous function $[a,b]$.

ii) $f(x) \geq 0$ for all x in $[a,b]$

iii) $a \leq b$

Then,

$\displaystyle\int_a^b f(x)\,dx$ = the area under the graph of f in $[a,b]$



"Def" (definite integral):

To calculate

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where}$$

$$F'(x) = f(x).$$

So F is antiderivative of f

Why?-ish

$$\int_a^b g'(x)\,dx = g(b) - g(a)$$

$$\int_a^b F'(x) = F(b) - F(a)$$

Approx.

$$\sum \underbrace{\underbrace{g'(x)}_{\text{height of rectangle}} \underbrace{\Delta x}_{\text{width of rectangle}}}_{\text{area of rectangle}} = g(b) - g(a)$$

Sum: Add up areas of rectangles

$$g'(x) \approx \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

Def. of derivative

$$\Delta x\, g'(x) \approx g(x+\Delta x) - g(x)$$

$$\sum_a^b g(x+\Delta x) - g(x) = g(b) - g(a)$$

"sum of lots of small changes = total change"

Could also write out sum and get cancelling. ▨

Ex: $\int_0^1 \underbrace{x^2}\,dx = [\overbrace{\tfrac{1}{3}x^3 + C}^{F(x)}]_{x=0}^{1}$

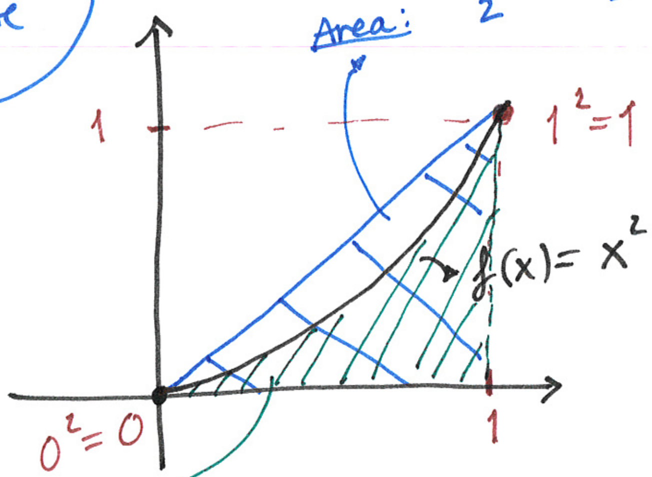
$\int_0^1 f(x)dx = F(1) - F(0)$ where F is an antiderivative of f.

$$= \underbrace{\left(\tfrac{1}{3}1^3 + C\right)}_{F(1)} - \underbrace{\left(\tfrac{1}{3}0^3 + C\right)}_{F(0)}$$

$$= \frac{1}{3} + \cancel{C} - \cancel{C} = \frac{1}{3}$$

Cancellation of constant always happens: Won't write down from now

Figure:



Area : $\int_0^1 x^2 dx = \frac{1}{3} < \frac{1}{2} =$ Area of blue triangle

Q: $\int_1^2 \overbrace{\ln(x)}^{f(x)}\,dx = [\overbrace{x\ln(x) - x}^{F(x)}]_{x=1}^{2}$

TRICK:

$\ln(x) = 1 \cdot \ln(x)$

$\int \underbrace{\ln(x)}_{v} \cdot \overset{u'}{1}\, dx = \quad \overset{uv}{x\ln(x)} - \overset{\int uv'dx}{\int x \tfrac{1}{\cancel{x}} dx} = \underline{x\ln(x) - x + C}$

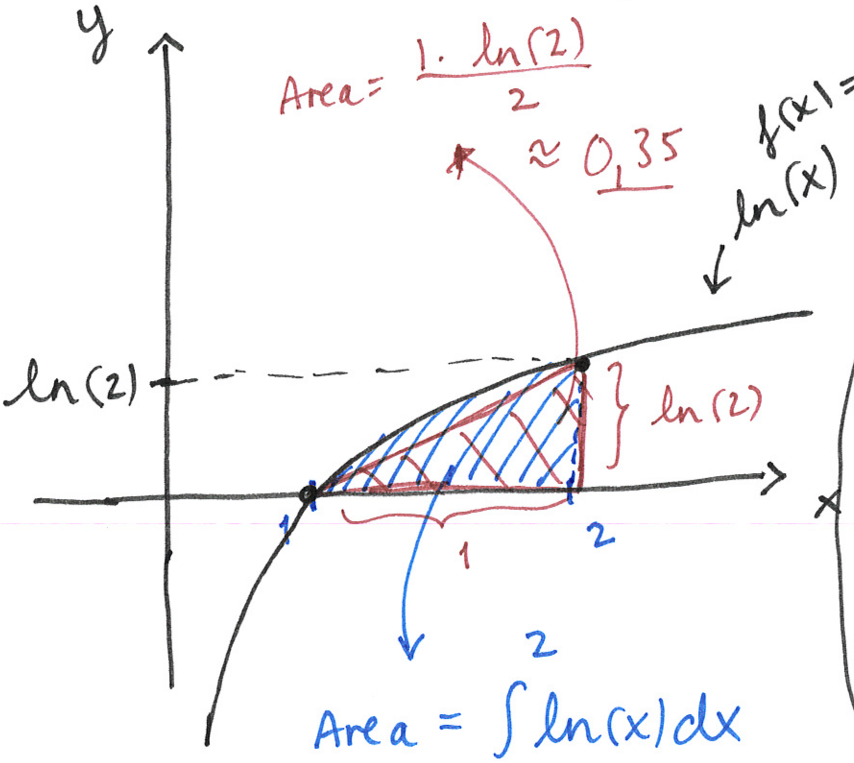
$u' = 1 \Rightarrow u = x$
$v = \ln x \Rightarrow v' = \frac{1}{x}$

$$= \underbrace{(2\ln(2) - 2)}_{F(2)} - \underbrace{(1\ln(1) - 1)}_{F(1)}$$

$\ln(1) = 0$

$$= 2\ln(2) - 2 + 1 = \underline{\underline{2\ln(2) - 1}}$$

$$(\approx 0{,}386)$$



To draw $\ln(x)$:

$\ln(1) = 0$

$\ln(2) > 0$

$\lim_{x \to \infty} \ln(x) = \infty$

$\lim_{x \to 0^+} \ln(x) = -\infty$

$(\ln x)' = \frac{1}{x} > 0, \quad x > 0$

ln is increasing, slower and slower

$$\text{Area} = \int_1^2 \ln(x)\,dx$$

$$\approx 0{,}386 > 0{,}35$$

$\approx$ Area of red triangle

$$\underline{Ex}: \int_{x=0}^{x=1} x\sqrt{x^2+1}\,dx = \int_{u=1}^{u=2} \cancel{x}\sqrt{u}\,\frac{1}{2\cancel{x}}\,du$$

$u = x^2 + 1$

$du = 2x\,dx$

$dx = \frac{1}{2x}\,du$

$x = 0 \Rightarrow u = 0^2 + 1 = 1$

$x = 1 \Rightarrow u = 1^2 + 1 = 2$

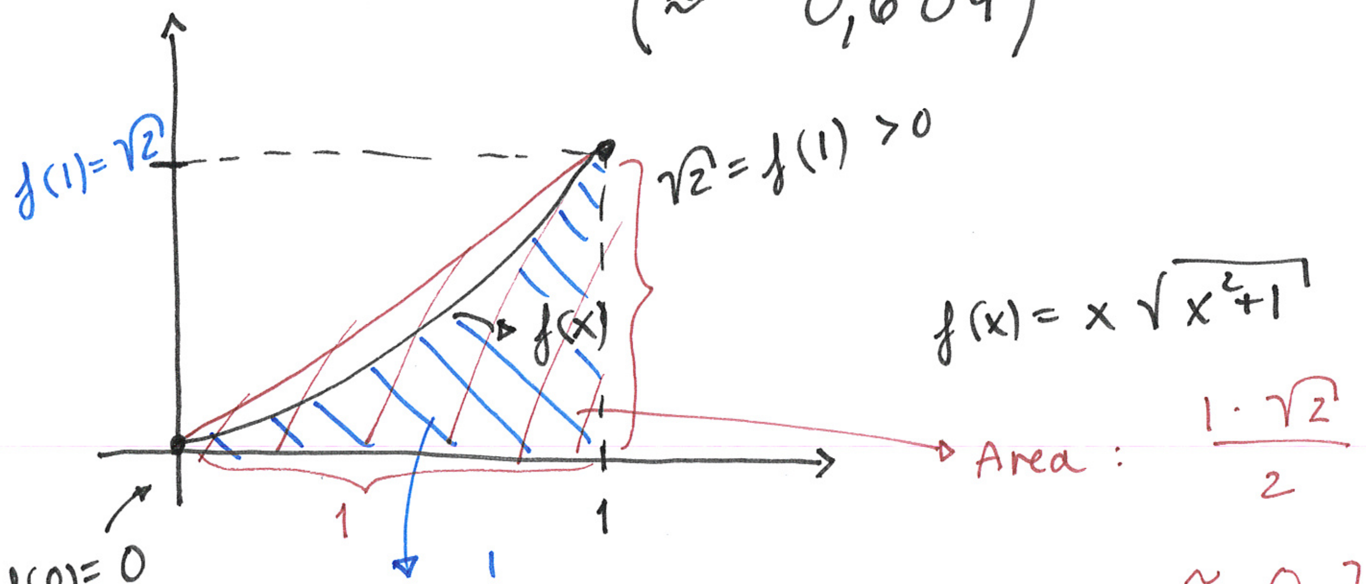
$$= \int_1^2 \frac{1}{2}\underbrace{\sqrt{u}}_{u^{\frac{1}{2}}}\,du$$

$$= \left[\frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{u=1}^{2} = \left[\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{\frac{3}{2}} \right]_{u=1}^{2}$$

$$u^{1+\frac{1}{2}} = u^{1} u^{\frac{1}{2}} = u\sqrt{u}$$

$$= \frac{2\sqrt{2}}{3} - \frac{1\sqrt{1}}{3} = \underline{\underline{\frac{1}{3}\left(2\sqrt{2}-1\right)}}$$

$$(\approx 0{,}609)$$



$$\approx 0{,}7$$

$$\text{Area} = \int_0^1 f(x)\,dx \approx 0{,}609$$

$$\boxed{0{,}609 < 0{,}7}$$

<u>NOTE</u>: Mind the integration bounds when doing substitution!

<u>Alternative</u>: Instead of inserting into:

$$\left[\frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] = \left[\frac{1}{2} \cdot \frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{x=0}^{x=1}$$

sub. back in for x

NB!

Theorem: If f is a continuous function on $[a, b]$ such that $f(x) \geq 0$ for x in $[a, b]$, then

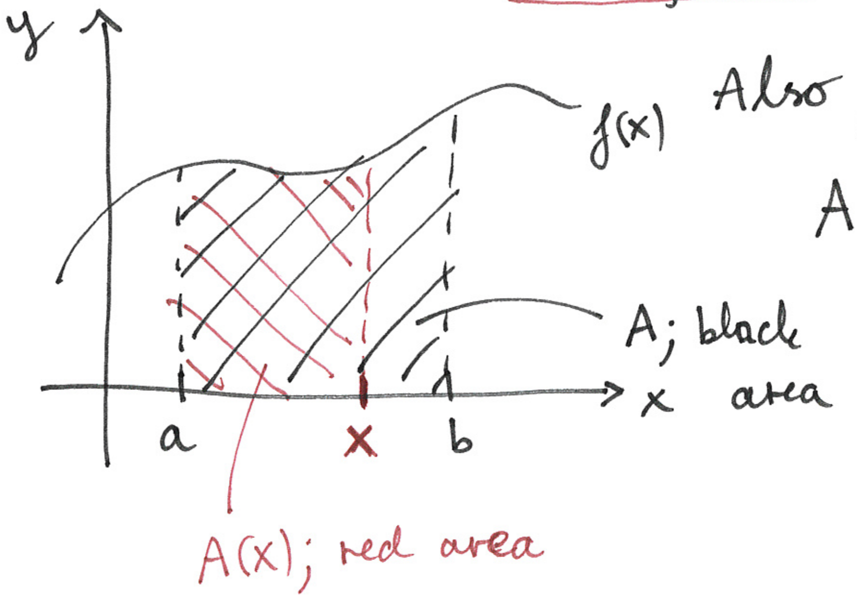
the area under the graph of $f(x)$ in the interval $[a,b]$ $\overset{\text{Will "prove"}}{=} \displaystyle\int_a^b f(x)\,dx \overset{\text{Already "show"}}{=} F(b) - F(a)$

where $F'(x) = f(x)$, so F is an anti-derivative of f.

Why? "Proof": of the first equality

Define

$$\underbrace{A(x)}_{\text{Area function}} = \text{the area under } y = f(x) \text{ for } [a, x]$$



Also, let

$$A = \text{area under } y = f(x) \text{ in } [a, b]$$

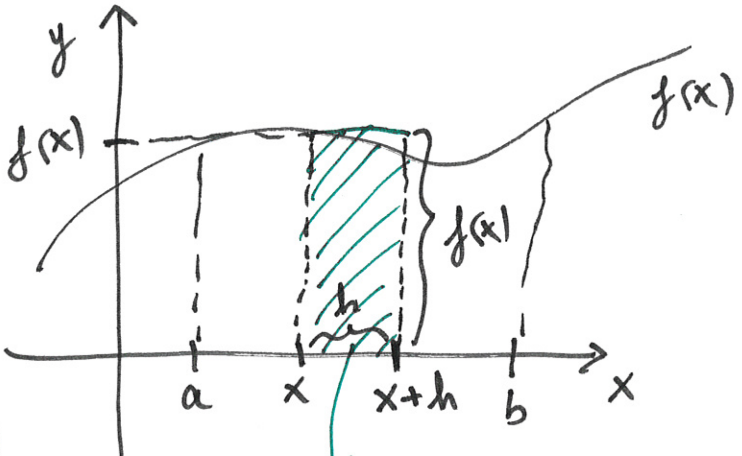
FACTS:

- $A(a) = 0$
- $A(b) = A$

$$A'(x) \approx \frac{A(x+h) - A(x)}{h}$$

$$= \frac{\text{area of strip}}{h}$$

$$= \frac{f(x)\cdot \cancel{h}}{\cancel{h}} = \underline{f(x)}$$



Area : $\boxed{A(x+h) - A(x)}$

<u>So:</u> $A'(x) \approx f(x)$, hence $A(x)$ is an anti-derivative of $f(x)$. But then,

$$\int_a^b f(x)\,dx = \Big[A(x)\Big]_{x=a}^{b} = A(b) - A(a)$$

$$= A - 0 = \underbrace{A}_{\text{area under graph between a and b}}$$

$\boxed{\text{FACTS}}$

Improper integrals

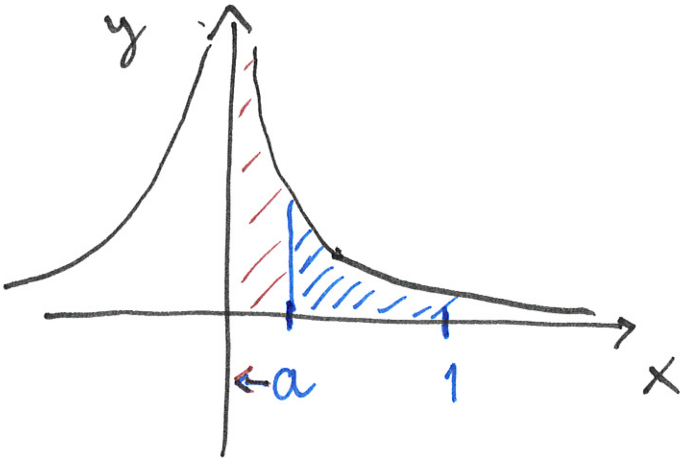
<u>What if:</u> 1) $f(x)$ is not continuous on $[a,b]$?

OR

2) $a = -\infty$ or $b = \infty$?

<u>Ex:</u> $$\int_0^1 \underbrace{\frac{1}{x^2}}_{} dx \overset{\text{THIS MEANS:}}{=} \lim_{a\to 0} \int_a^1 \frac{1}{x^2}\,dx$$

$\frac{1}{x^2}$ is not defined for $x=0$



$$\lim_{x \to 0^{\pm}} \frac{1}{x^2} = \infty$$

$$\lim_{x \to \pm\infty} \frac{1}{x^2} = 0$$

$$\int_a^1 \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_{x=a}^{1} = \left[-\frac{1}{x}\right]_{x=a}^{1}$$

$$= \underline{-1 + \frac{1}{a}},$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{x^2}\,dx$$

$$= \lim_{a \to 0^+} \left(-1 + \overset{\to\infty}{\frac{1}{a}}\right) = \underline{\underline{\infty}}$$