More linear systems: Number of solutions

EBA1180
Lect. 32
Spring 25

Ex:
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 17 \\ x_1 - 2x_2 - x_3 + 4x_5 &= 8 \\ 2x_1 + x_2 - 5x_3 + 7x_4 &= 11 \end{aligned}$$

Def (Pivot position): A pivot position is a position where there is a pivot in the echelon form.

Ex ctd:
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 1 & -2 & -1 & 0 & 4 & 8 \\ 2 & 1 & -5 & 7 & 0 & 11 \end{array}\right] \begin{array}{c} \\ -1 \\ -2 \end{array}$$

Add $(-1)*$ row 1 to row 2
Add $(-2)*$ row 1 to row 3

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -3 & -2 & -1 & 3 & -9 \\ 0 & -1 & -7 & 5 & -2 & -23 \end{array}\right]$$

switch rows 2 and 3

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & -7 & 5 & -2 & -23 \\ 0 & -3 & -2 & -1 & 3 & -9 \end{array}\right] \begin{array}{c} \\ \\ -3 \end{array}$$

add $(-3)*$ row 2 to row 3 $\sim$

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 17 \\ 0 & -1 & -7 & 5 & -2 & -23 \\ 0 & 0 & 19 & -16 & 9 & 60 \end{array}\right]$$

Echelon form!

Pivots! Pivot positions are $(1,1)$, $(2,2)$, $(3,3)$.

The linear system has two degrees of freedom (x_4, x_5 are free). Hence, the system has infinitely many solutions.

Why? From the echelon form:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 17 \\ -x_2 - 7x_3 + 5x_4 - 2x_5 &= -23 \\ 19x_3 - 16x_4 + 9x_5 &= 60 \quad \Rightarrow \end{aligned}$$

$$19x_3 = 60 + 16x_4 - 9x_5$$

$$\vdots$$

$x_2 = \ldots$ via x_4 and x_5 ...

$x_1 = \ldots$ via x_4 and x_5 ...

$\rightsquigarrow$ Can choose any x_4 and x_5 and the original linear system still holds.

Result: For any linear system, the pivot positions determine the number of solutions.

in echelon form

Different cases:

in extended matrix form

i) Pivot position in the last column: No solutions.

Ex:

$$\left[\begin{array}{ccc|c} \cdots & & \cdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{array}\right] \rightarrow$$

Says:

$0 = 1$; Not true! Never.

ii) No pivot position in the last column:

The linear system has solutions.

a) Pivot positions in all variable columns:

One solution.

Ex:

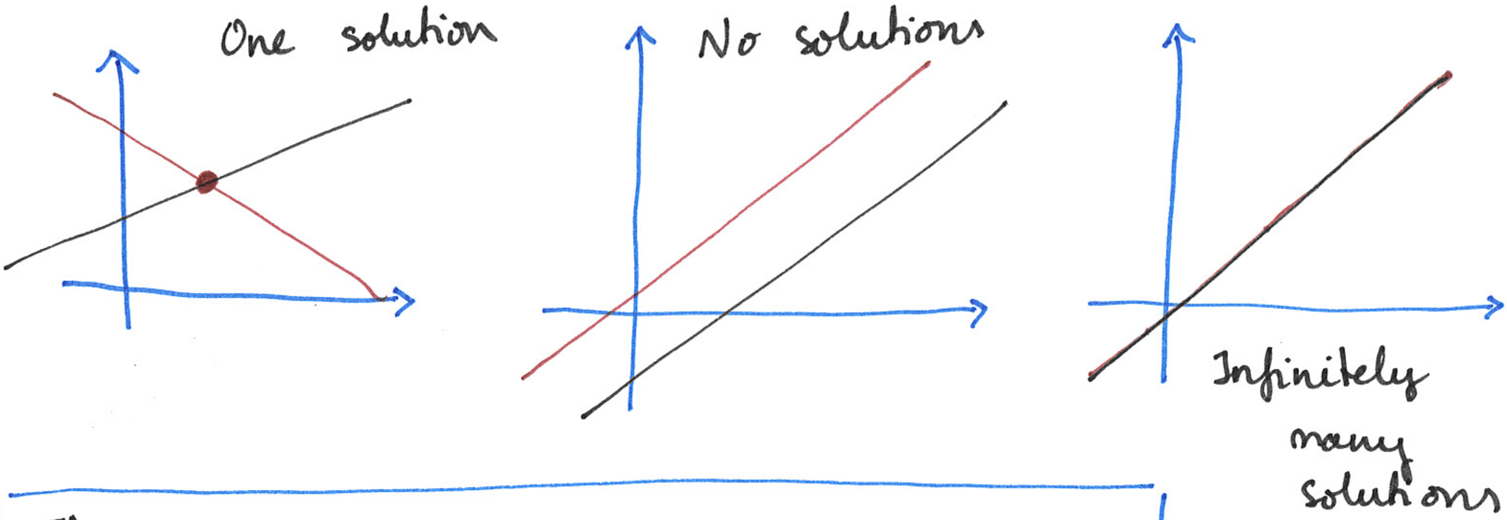
$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & \cdots & \cdots & \vdots \\ 0 & 1 & \cdots & \vdots \\ 0 & 0 & 1 & \vdots \end{array}\right] \end{array} \begin{array}{l} \rightarrow x_1 = \text{number} \\ \rightarrow x_2 = \text{number} \\ \rightarrow x_3 = \text{number} \end{array}$$

b) There are variable columns without pivot positions: Infinitely many solutions.

Ex:

$$\left[\begin{array}{cccc|c} 1 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \end{array}\right]$$

The columns without pivots correspond to free variables.



Theorem: Any linear system has either

i) No solutions ⇝ Inconsistent

ii) One unique solution

iii) Infinitely many solutions

} → Consistent

START 11.01

Computations with matrices and vectors

Def ($m \times n$ matrix): "m by n", "m times n"

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Ex:

$$A = \left[\begin{array}{ccc} 1 & 2 & 4 \\ 7 & -1 & 0 \end{array}\right] \Big\} \text{ 2 rows}$$

Capital letters for matrices

3 columns

Dimension: 2×3 matrix

Read: "2 by 3" or "2 times 3"

$$A = \begin{array}{cccc} \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] & \begin{array}{c} 1 \\ 2 \end{array} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} & \end{array}$$

a_{12}

column 2

row 1

- Addition: $A + B$
- Subtraction: $A - B$

Defined if A and B have the same dimension (e.g. both $m \times n$)

Result is a matrix of the same size as A (B)

- Scalar multiplication: $r \cdot A$ → Result will be a matrix of same size as A

Always defined

r : scalar (number)

A : matrix

Ex:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2+(-1) & 3+1 \\ -1+1 & 0+2 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

Do addition / subtraction position by position.

Ex:

$$2 \cdot \begin{bmatrix} 1 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 4 \\ 2\cdot(-1) & 2\cdot 2 \\ 2\cdot 0 & 2\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -2 & 4 \\ 0 & 2 \end{bmatrix}$$

Do multiplication by scalar position by position.

Def (n-vector)

An n-vector is a matrix with n rows and 1 column (a column vector).

- Write vectors as $\vec{v}$ = boldface $\mathbf{v} = \underline{v}$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; a 3-vector viewed as a column vector

Lower case letters for vectors

Vector operations

→ **Addition:** $\vec{v} + \vec{w}$

For vectors of same length

→ **Subtraction:** $\vec{v} - \vec{w}$

always defined

fixed number

→ **Scalar multiplication:** $r \cdot \vec{v}$ (r scalar)

Ex:

ADD:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2+(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}}$$

SUBTRACT:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 2-(-1) \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ 3 \end{bmatrix}}}$$

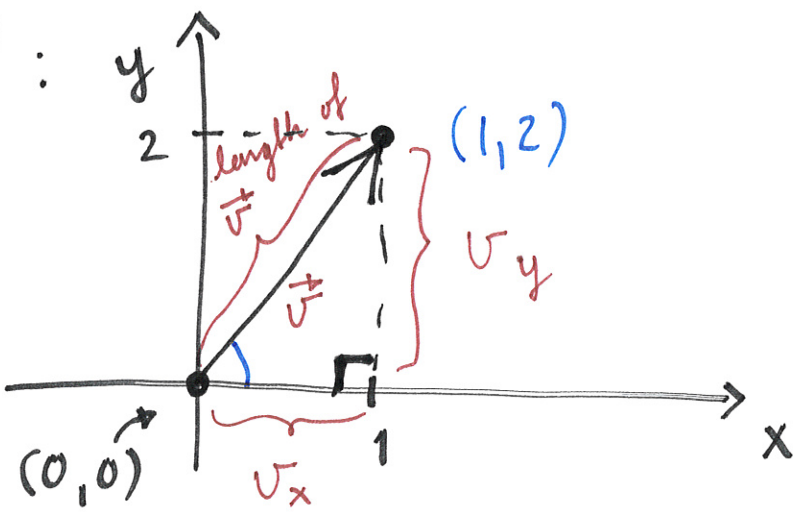
SCALAR MULTIPLICATION:

$$2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}}$$

Geometric interpretation of vectors

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

Corresponds to an arrow from $(0,0)$ to (v_x, v_y): $= (1,2)$



A vector has length (magnitude) and a direction.