Examples

EBA1180

Lect. 36

Spring 25

to be determined in model

x, y, z : variables

r : parameter

given from outside

Ex:

3x3 linear system

$$\begin{cases} rx + 2y - z = 3 \\ x + (r+1)y - z = 3 \\ -x - 2y + rz = 1 - r \end{cases}$$

$$|A| = \begin{vmatrix} r & 2 & -1 \\ 1 & r+1 & -1 \\ -1 & -2 & r \end{vmatrix}$$

coefficient matrix

Sign of cofactor expansion 3x3:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= r \begin{vmatrix} r+1 & -1 \\ -2 & r \end{vmatrix}$$

$$- 2 \begin{vmatrix} 1 & -1 \\ -1 & r \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & r+1 \\ -1 & -2 \end{vmatrix}$$

$$= r((r+1)r - 2) - 2(r-1) - 1(-2 + r + 1)$$

$$= r(r^2 + r - 2) - 2r + 2 + 2 - r - 1$$

$$= r(r+2)(r-1) - 3r + 3$$

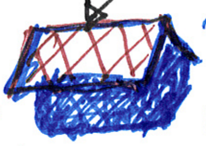
abc-formula etc. to factorize

$$= r(r+2)\underline{(r-1)} - 3\underline{(r-1)}$$

$$= (r-1)(r(r+2) - 3)$$

$$= (r-1)(r^2+2r-3)$$

$$= (r-1)(r-1)(r+3) = \underline{\underline{(r-1)^2(r+3)}}$$



2 cases:

1) $|A| = 0$ → Either no solutions or infinitely many → Which one?

2) $|A| \neq 0$: Later.

$$(r-1)^2(r+3) = 0$$

$$\underline{r=1}, \quad \underline{r=-3}$$

i) $r=1$: Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 2 & -1 & 3 \\ -1 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon form!

→ $0 = 3$ NOT TRUE!

$\Rightarrow$ No solutions for $r=1$.

ii) $r=-3$:

$$\left[\begin{array}{ccc|c} -3 & 2 & -1 & 3 \\ 1 & -2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -3 & 2 & -1 & 3 \\ -1 & -2 & -3 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & -4 & -4 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 0 & -4 & -4 & 12 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

-1

$\Rightarrow 0 = -5$ NOT TRUE!

$\Rightarrow$ No solutions for $r = -3$.

2) $|A| \neq 0$: $(r-1)^2(r+3) \neq 0$

$r \neq 1$, $r \neq -3$ $\Rightarrow$ Unique solution

Find this solution via Cramer's rule: $\rightarrow$ Easier than Gaussian elimination due to parameters

To find x:

Know: $|A| = (r-1)^2(r+3)$

Need: $|A_1(\vec{b})| = \begin{vmatrix} 3 & 2 & -1 \\ 3 & r+1 & -1 \\ 1-r & -2 & r \end{vmatrix}$

$\begin{bmatrix} 3 \\ 3 \\ 1-r \end{bmatrix}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= 3\begin{vmatrix} r+1 & -1 \\ -2 & r \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 1-r & r \end{vmatrix} + (-1)\begin{vmatrix} 3 & r+1 \\ 1-r & -2 \end{vmatrix} = \dots = \underline{\underline{2r^2 - r - 1}}$$

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From Cramer's rule :

$$x = \frac{|A_1(\vec{b})|}{|A|} = \frac{2r^2 - r - 1}{(r-1)^2(r+3)}$$

$$= \frac{(2r+1)\cancel{(r-1)}}{(r-1)^{\cancel{2}}(r+3)}$$

abc-formula

Complete square etc.

$$= \underline{\underline{\frac{2r+1}{(r-1)(r+3)}}}, \quad r \neq 1, -3$$

Can find y and z similarly.

$|A_2(\vec{b})|$ $\qquad$ $|A_3(\vec{b})|$

Vector equations

$$x \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}}_{\vec{v}_1} + y \underbrace{\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}_{\vec{v}_2} + z \underbrace{\begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix}}_{\vec{v}_3} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}}_{\vec{b}}$$

A vector equation

$$\underbrace{x}_{c_1}\vec{v}_1 + \underbrace{y}_{c_2}\vec{v}_2 + \underbrace{z}_{c_3}\vec{v}_3 = \vec{b}$$

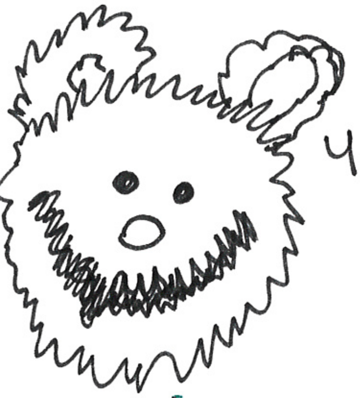
$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$$

Can $\vec{b}$ be written as a lin. comb. of $\vec{v}_1, \vec{v}_2, \vec{v}_3$? How?

$$\begin{bmatrix} x \\ 0 \\ 4x \end{bmatrix} + \begin{bmatrix} 3y \\ -y \\ 2y \end{bmatrix} + \begin{bmatrix} 4z \\ -z \\ 6z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x + 3y + 4z \\ -y - z \\ 4x + 2y + 6z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x + 3y + 4z &= 3 \\ -y - z &= 1 \\ 4x + 2y + 6z &= 2 \end{aligned}$$



Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 4 & 2 & 6 & 2 \end{array}\right] -4 \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & -10 & -10 & -10 \end{array}\right] -10$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -20 \end{array}\right] \Rightarrow \quad 0 = -20$$

NOT TRUE $\Rightarrow$ No solutions!

Inverse matrices

Def (Inverse matrix): Let A be an $n \times n$ matrix. square

An inverse of A is a matrix A^{-1} $n \times n$ s.t.

such that

$$\boxed{\begin{array}{c} A \cdot A^{-1} = I \quad \underline{\text{and}} \\ A^{-1} \cdot A = I \end{array}}$$

$$2^{-1} = \frac{1}{2}$$

$$2 \cdot 2^{-1} = 2 \cdot \frac{1}{2} = 1$$

$$2^{-1} \cdot 2 = \frac{1}{2} \cdot 2 = 1$$

<u>Ex</u>: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $|A| = 4 - 1 = 3 \neq 0$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \cdot \underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\text{some other matrix}}$$

$$= \frac{1}{3}\begin{bmatrix} 4-1 & -2+2 \\ 2-2 & -1+4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

<u>Can check (DIY):</u>

$$\underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{\text{the other matrix}} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Hence (from def.): $\underline{\underline{A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}}$

FORMULA (Inverse, $n=2$):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$|A| = ad - bc \neq 0$:

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$|A| = ad - bc = 0$:

No inverse of A.

FACTS: i) The inverse of A does not always exist.

Actually, A is invertible (i.e., A^{-1} exists) if and only if $|A| \neq 0$.

ii) If A has an inverse, it is unique.

iii) General formula for A^{-1} when $|A| \neq 0$:

Transpose

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^{T}$$

where C_{ij} are the cofactors.