Inverse matrices

EBA1180 Spring25 Sect. 37

Ex: $2x + y = 4$

$x + 2y = 3$

Matrix form:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\vec{b}}$$

$A\vec{x} = \vec{b}$

If A^{-1} exists:

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\boxed{\vec{x} = A^{-1}\vec{b}}$$

Advantage of A^{-1}: Quickly solve lin. syst. with many different r.h.s.

$$\vec{x} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8-3 \\ -4+6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{3} \end{bmatrix}$$

So: $(x, y) = \left(\frac{5}{3}, \frac{2}{3}\right)$

Computing inverse matrices

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

$$c_{ij} = (-1)^{i+j} M_{ij}$$

$$|A| = 1\,(\overset{6}{18 - 12}) - 1\,(\overset{5}{9 - 4}) + 1\,(\overset{1}{3 - 2})$$

$= 2 \neq 0$ so A has an inverse.

Use formula to find A^{-1}:

$$A^{-1} = \frac{1}{|A|} \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{23} \end{bmatrix}^T}_{\text{Adjoint matrix of } A:\ adj(A)}$$

Compute c_{ij}'s: $c_{11} = 6$ $\quad c_{12} = -5$ $\quad c_{13} = 1$

$c_{21} = -6$ $\quad c_{22} = 8$ $\quad c_{23} = -2$

$c_{31} = 2$ $\quad c_{32} = -3$ $\quad c_{33} = 1$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}^T$$

$$= \frac{1}{2} \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

Alternative way of finding A^{-1}:

A $n \times n$ matrix

$$[A \mid I] \sim \dots \sim [B \mid C]$$

Reduced echelon form:
Aim for $B = I$

2 cases:

$B = I$: $A^{-1} = C$

$B \neq I$: A^{-1} doesn't exist.

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, seen before:

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Alternative method:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \Bigg| \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] -2$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{array}\right] \mid : (-3) \sim \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right] -2$$

Echelon form

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

Reduced echelon form

$$= [\, I \mid A^{-1} \,], \text{ so}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Why does this work? 3×3

$$A = \begin{bmatrix} 1 & \cdots \\ 2 & \cdots \\ \cdots & \end{bmatrix} \underset{\text{1st elementary row op.}}{\overset{-2}{\sim}} \begin{bmatrix} 1 & \cdots \\ 0 & \cdots \\ \cdots & \end{bmatrix} = B, \text{ then}$$

$$B = E_1 \cdot A$$

elementary matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

the matrix found by doing the elementary row operation on the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{-2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

Each row operation corresponds to matrix multiplication

START: 13.00

$$[A \mid I] \underset{\text{row op. 1}}{\sim} \cdots \underset{\text{row op. r}}{\sim} [B \mid C]$$

$$\Updownarrow$$

$$\underbrace{E_r}_{\text{row op. r}} \cdots \underbrace{E_2}_{\text{row op. 2}} \underbrace{E_1}_{\text{row op. 1}} A = B$$

Multiply with one elementary matrix for each row op.

and

$$E_r \cdots E_2 E_1 I = C$$

Then, if $B = I$:

$$\overbrace{E_r \cdots E_2 E_1}^{C} A = I$$

$$\underbrace{E_r \cdots E_2 E_1}_{C} = C$$

So:

$$\underbrace{C}_{A^{-1}} A = I$$

If $B \neq I$: $|A| = 0 \Rightarrow$ No inverse.

Will have 0 in diagonal $\Rightarrow |B| = 0$.

But $|A| = \underbrace{|B|}_{0} \cdot \text{scaling} \cdot \text{switch sign}$

$\Rightarrow |A| = 0$

Inner product (dot product) of vectors

Def (Inner product): Let $\vec{v}, \vec{w}$ be n-vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}. \quad \text{Then,}$$

$$\vec{v} \underset{\text{dot}}{\bullet} \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

Ex: $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$\vec{v} \bullet \vec{w} = 1 \cdot 3 + 2 \cdot (-1) = 3 - 2 = \underline{\underline{1}}$$

Read: "$\vec{v}$ dot $\vec{w}$"

Exercise sheet 35

7) b), c), d)

Profit per stock:

	A (x)	B (y)	C (z)
1	20	5	30
2	40	-50	180
3	-20	25	-265

Profit from owning 1 unit of stocks A, B, C resp. in scenario 3

↳ Profit for owning 1 unit of stock A in each of the 3 scenarios

Budget constraint: $60x + 75y + 320z = C$

400000

Which  profits are possible?

Profits: (R_1, R_2, R_3)

Profit in scenario 1: $20x + 5y + 30z = R_1$

Profit in scenario 2: $40x - 50y + 180z = R_2$

Profit in scenario 3: $-20x + 25y - 265z = R_3$

+ budget constraint: Gaussian elimination:

$$\left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 40 & -50 & 180 & R_2 \\ -20 & 25 & -265 & R_3 \\ 60 & 75 & 320 & C \end{array}\right] \begin{array}{c} \\ -2 \\ 1 \\ -3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 30 & -235 & R_3 + R_1 \\ 0 & 60 & 230 & C - 3R_1 \end{array}\right] \begin{array}{c} \\ \\ \frac{1}{2} \\ 1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 0 & -175 & R_3 + R_1 + \frac{1}{2}(R_2 - 2R_1) \\ 0 & 0 & 350 & C - 3R_1 + (R_2 - 2R_1) \end{array}\right] \begin{matrix} \\ \\ \\ 2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 20 & 5 & 30 & R_1 \\ 0 & -60 & 120 & R_2 - 2R_1 \\ 0 & 0 & -175 & R_3 + \frac{1}{2} R_2 \\ 0 & 0 & 0 & C - 5R_1 + 2R_2 + 2R_3 \end{array}\right]$$

For system to have a solution, must have:

$$C - 5R_1 + 2R_2 + 2R_3 = 0$$