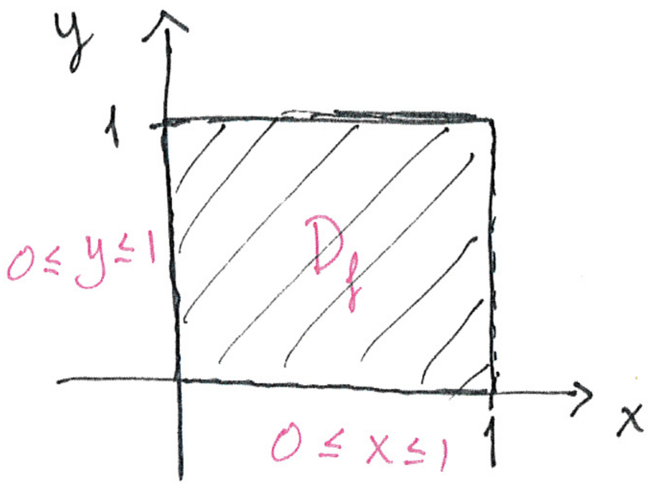
Warm up: $f(x,y) = x + y$, $\underbrace{0 \leq x, y \leq 1}_{D_f}$

What are the (global) max / min points?



See directly:

Maximum: $(1,1) \Rightarrow$
$f(1,1) = 2$

Minimum: $(0,0) \Rightarrow$
$f(0,0) = 0$

NOTE: f has both (global) min and max.

Q1: What if D_f : $0 < x, y < 1$

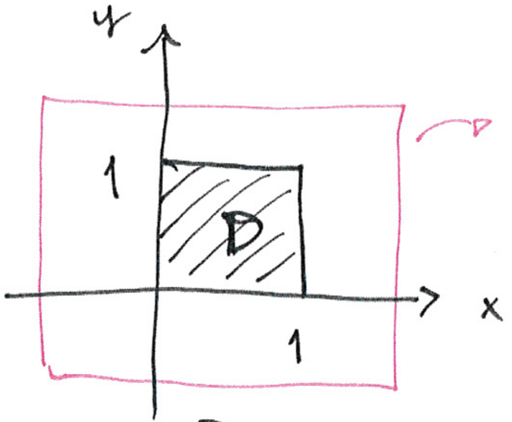
Q2: What if min/max $x+y$ over all of $\mathbb{R}^2$?

Constrained optimization and the extreme value theorem

- $f(x,y)$ is a continuous function on a set D in $\mathbb{R}^2$.

Def (Bounded set): A subset D of $\mathbb{R}^2$ is bounded if there exists a rectangle in $\mathbb{R}^2$ (with finite side lengths) that includes all of D.

Ex:



Rectangle containing D

$\Rightarrow$ Bounded ✓

Closed ✓

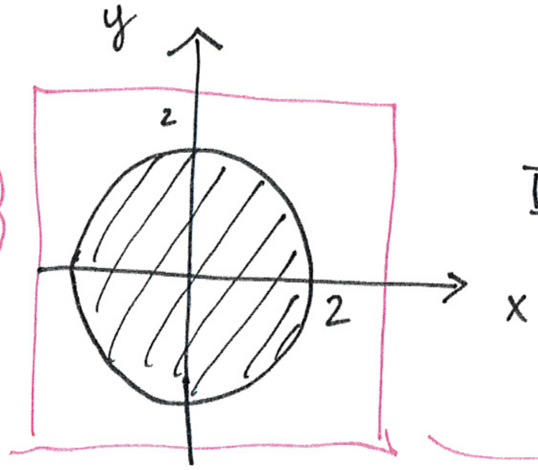
$\Downarrow$

Compact ✓

$D: \; 0 \le x, y \le 1$

Ex:

losed? Bounded? ompact?



Q:

$D: \; x^2 + y^2 \le 4$

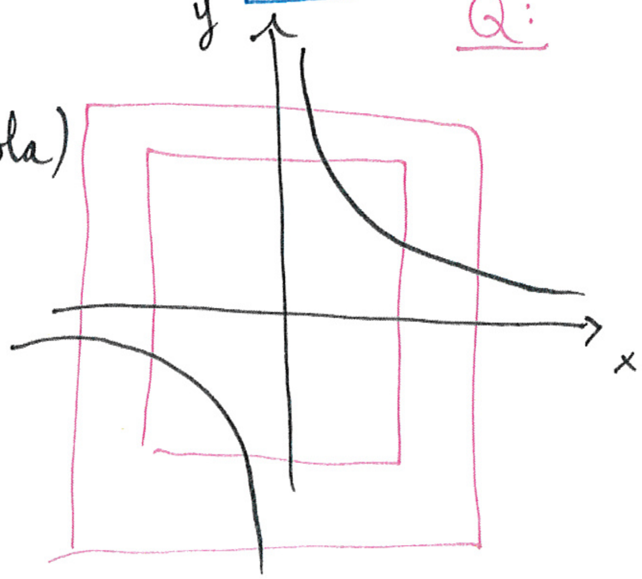
Closed? ✓ (includes boundary)

Bounded? ✓

$\Downarrow$

$\boxed{\text{Compact}}$

Q:



Ex: $D: \; xy = 1$ (hyperbola)

$y = \dfrac{1}{x}$

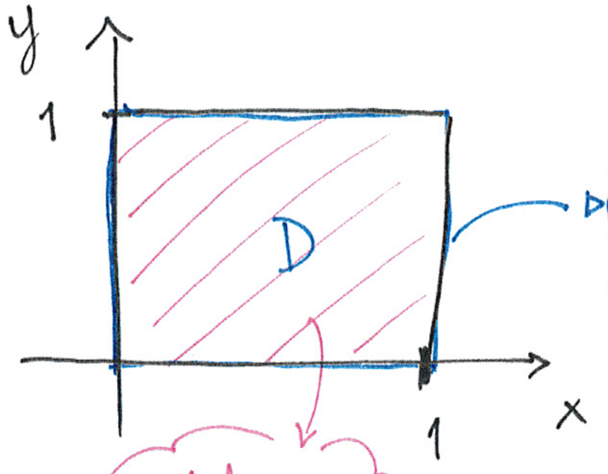
Closed? ✓

Bounded? NO!

The boundary: ∂D

Ex: max/min $f(x,y) = x^2 + y^2$ when

$0 \le x, y \le 1$



∂D = the boundary of D (the four sides of the square)

Interior of D

Max: Make x and y as large as possible $\Rightarrow$

$x = y = 1 \Rightarrow f(1,1) = 2$

Q:

Min: Make x and y as small as possible $\Rightarrow$

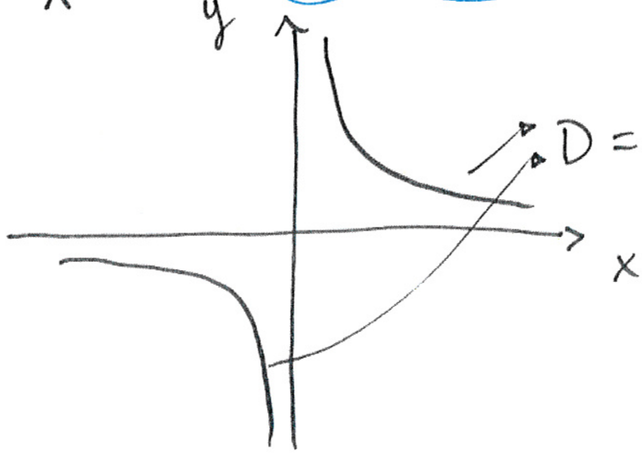
$x = y = 0 \Rightarrow f(0,0) = 0$

Ex: D:

$xy = 1$

$y = \frac{1}{x}$

$x = 0$ impossible since $xy = 1$



$D = \{(x,y) \in \mathbb{R}^2 : xy = 1\}$

∂D = boundary of D

= all pts. on D.

EVT: • f continuous? OK!

• D compact? Closed and bounded?
Yes! ✓ ✓

$\Rightarrow$ $\boxed{f(x,y) \text{ has (global) max and min}}$

EVT holds

$\Downarrow$

There is a max and min among the candidate points.

What is the max/min? Compare values of candidate points.

i) Stationary: $(0,0) \Rightarrow f(0,0) = 0$

ii) Critical: None.

$f(x,y) = x^2 + y^2$

iii) Boundary:

A: $f(1,y) = 1 + y^2$, $-1 \leq y \leq 1$, see dir.:

max: $f(1,1) = f(1,-1) = 2$,

min: $f(1,0) = 1$

The lowest value among the candidate pts. is:

$$f_{min} = 0 \rightarrow \text{global min. value}$$

at the min. pt. $(0,0)$

Alt. method side A: $f(1,y) = 1 + y^2, \; -1 \le y \le 1$

$x = 1 \Rightarrow$

$(1+y^2)' = 2y = 0 \Rightarrow y = 0$

Candidates: $y=0, \; y=-1, \; y=1$

What is this? A cup where we slice through with a square column: Get min. at bottom of cup & max at corners of square column.