Lagrange problems

EBA 1180
Spring 25
Lect. 45

→ **Lagrange problems**: Optimization problems (max/min) with equality constraints

(★) $\boxed{\max/\min \ f(x,y) \quad \text{when} \quad \underbrace{g(x,y)}_{\text{function}} = \underbrace{a}_{\text{constant}}}$

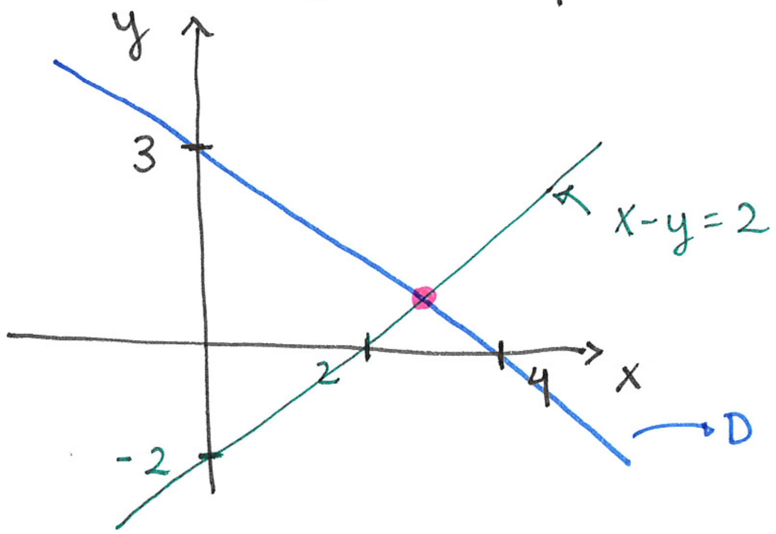
Ex: $\min f(x,y) = x^2 + y^2$ when $\underbrace{3x + 4y}_{g(x,y)} = \underbrace{12}_{a}$

Draw: $4y = 12 - 3x \quad | : 4$

$$y = 3 - \frac{3}{4}x$$

$x = 0$: $y = 3 - 0 = 3$

$y = 0$: $x = 4$



Ex: $\min f(x,y) = x^2 + y^2$ when $3x + 4y = 12$ and $x - y = 2$

$$\boxed{y = x - 2}$$

A point: Intersection of the two straight lines

Recap: General method:

1) Find candidate points.

i) Interior stationary points: NONE
ii) Other interior critical pts: NONE
iii) Boundary points: All admissible pts.

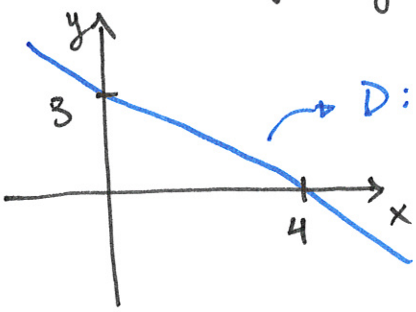
2) Determine whether any of these are max/min.

Extreme value theorem: If D is compact (closed and bounded) and f is continuous, then f has a max and min on D

Always true for Lagrange problems: = constraint

Not necessarily true for Lagrange problems

Ex: min $f(x,y) = x^2 + y^2$ when $3x + 4y = 12$



D: $3x + 4y = 12$; D is closed, but not bounded

$\Rightarrow$ D is not compact $\Rightarrow$ EVT can't be used.

Method of Lagrange multipliers

$$\underbrace{\mathcal{L}(x,y;\lambda)}_{\text{Lagrangian (Lagrange function)}} = f(x,y) - \lambda\underbrace{(g(x,y) - a)}$$

Lagrange multiplier: variable

NB: $= 0$ if the constraint in (★) holds

Ex: $= x^2 + y^2 - \lambda(3x + 4y - 12)$

Candidates for max/min : The stationary points of $\mathcal{L}$:

First order conditions

$$\text{FOC:} \begin{cases} \mathcal{L}'_x = f'_x - \lambda g'_x = 0 = 2x - 3\lambda \\ \mathcal{L}'_y = f'_y - \lambda g'_y = 0 = 2y - 4\lambda \end{cases}$$

Ex.

$$\text{C:} \left\{ \mathcal{L}'_\lambda = -(g(x,y)-a) = 0 = -(3x+4y-12) \right.$$

Constraint

$$g(x,y)-a=0 \qquad \Updownarrow$$

$$\boxed{g(x,y)=a} \qquad \boxed{3x+4y=12}$$

The constraint

Lagrange conditions: $\boxed{\text{FOC} + \text{C}}$

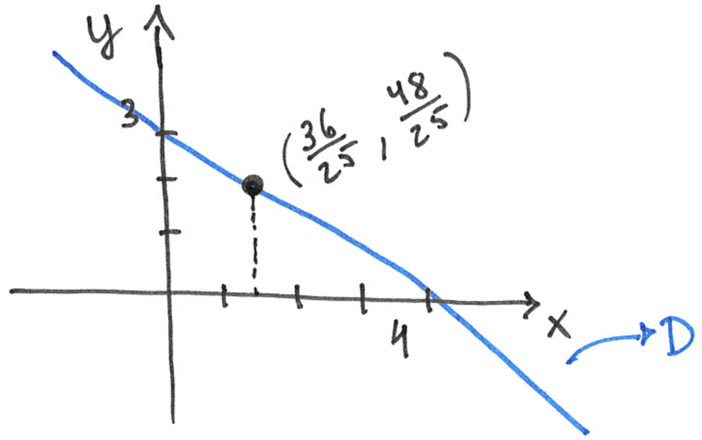
$$\begin{array}{lll} \text{FOC:} & \mathcal{L}'_x = & 2x - 3\lambda = 0 \quad : (1) \\ & \mathcal{L}'_y = & 2y - 4\lambda = 0 \quad : (2) \\ \text{C:} & & 3x + 4y = 12 \quad : (3) \end{array}$$

System of 3 eqns. and 3 unknowns: x, y, λ

To solve: Gaussian elimination or isolating variables + substituting.

$$\rightsquigarrow \lambda = \frac{24}{25}, \quad x = \frac{36}{25}, \quad y = \frac{48}{25}$$

Only one candidate point: $\left(\underbrace{\frac{36}{25}}_{x}, \underbrace{\frac{48}{25}}_{y} ; \underbrace{\frac{24}{25}}_{\lambda}\right)$



Alternative method: substitution

$$\boxed{\min f(x,y) = x^2 + y^2 \text{ when } \underbrace{3x + 4y = 12}_{y = 3 - \frac{3}{4}x}}$$

$$x^2 + y^2 = x^2 + \left(3 - \frac{3}{4}x\right)^2$$

$$= x^2 + 9 - \frac{9}{2}x + \frac{9}{16}x^2$$

$$= \frac{25}{16}x^2 - \frac{9}{2}x + 9 =: g(x)$$

Define a one-var. func. $g(x)$

$$\boxed{\underline{\text{Alt}}: \quad \min g(x) = \frac{25}{16}x^2 - \frac{9}{2}x + 9}$$

A one-var. optimization problem!

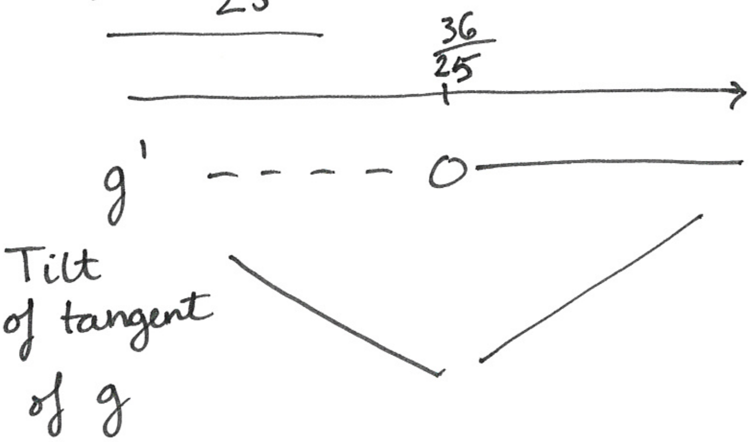
$$g'(x) = \frac{25}{8}x - \frac{9}{2} = 0 \quad \Big| \cdot 8$$

$$25x - 36 = 0$$

$$\underline{x = \frac{36}{25}}$$

$$y = 3 - \frac{3}{4} \cdot \frac{36}{25}$$

$$= \ldots = \underline{\frac{48}{25}}$$



Hence, $x = \frac{36}{25}$ is a $\underline{\underline{\text{minimum}}}$ for g.

Intuition: Lagrange multiplier method

Ex: max/min $f(x,y) = x^2 + y^2$ when $3x + 4y = 12$

Level curves of f: $f(x,y) = c$

$x^2 + y^2 = c$

$c > 0$: Circle, center $(0,0)$, $r = \sqrt{c}$

$c = 0$: A point $(0,0)$

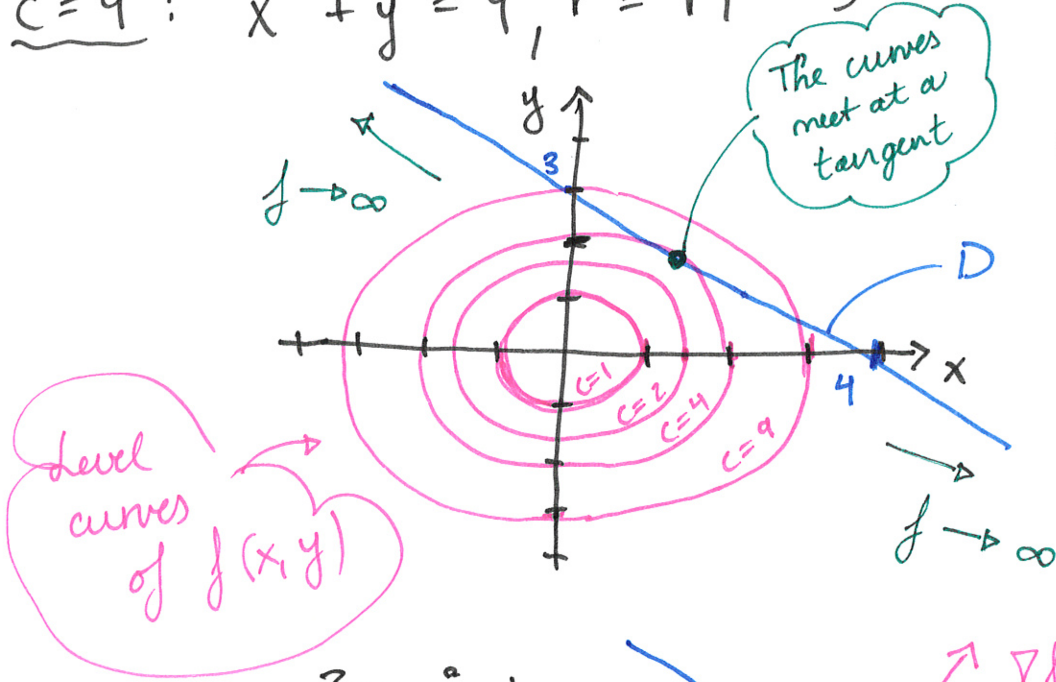
$c < 0$: No level curve

$c=1$: $x^2 + y^2 = 1$, $r = \sqrt{1} = 1$

$c=2$: $x^2 + y^2 = 2$, $r = \sqrt{2}$

$c=4$: $x^2 + y^2 = 4$, $r = \sqrt{4} = 2$

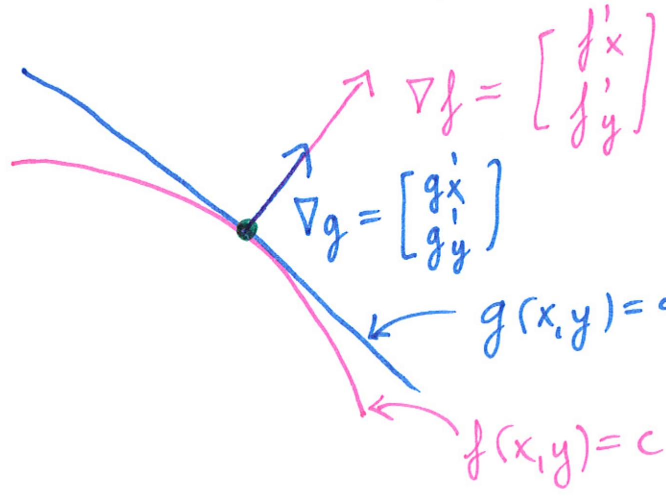
$c=9$: $x^2 + y^2 = 9$, $r = \sqrt{9} = 3$



(Should be circles! Poor scaling - sorry)

Zoom in:

Fig. 1:



also a level curve for g at level a

Candidates for max/min:

Points where the two curves meet at a tangent

$$\begin{cases} 3x + 4y = 12 : D \\ x^2 + y^2 = c : \text{level curve of } f \end{cases}$$

Slopes of tangents of level curves should be equal:

$$-\frac{f'_x}{f'_y} = -\frac{g'_x}{g'_y}$$

Notes on tangents of level curves: Implicit diff.

$$-\frac{2x}{2y} = -\frac{3}{4}$$

$\vdots$ (solve)

$$y = \frac{4}{3}x$$

Constraint: $3x + 4 \cdot \frac{4}{3}x = 12$

$\vdots$ (solve)

$$\underline{\underline{x = \frac{36}{25}}}$$

NOTE:

$$\nabla f = \lambda \nabla g$$

From Fig 1

Gradient of f is a scalar multiple of the gradient of g

$$\begin{bmatrix} f'_x \\ f'_y \end{bmatrix} = \lambda \begin{bmatrix} g'_x \\ g'_y \end{bmatrix}$$

$$\begin{cases} f'_x = \lambda g'_x \\ f'_y = \lambda g'_y \end{cases} \Rightarrow \begin{cases} \mathcal{L}'_x = f'_x - \lambda g'_x = 0 \\ \mathcal{L}'_y = f'_y - \lambda g'_y = 0 \end{cases}$$

Theorem: If (x^*, y^*) is a max/min in a Lagrange problem:

$$\boxed{{}^{\max}/_{\min}\ f(x,y) \text{ with } g(x,y) = a}$$

Then, either

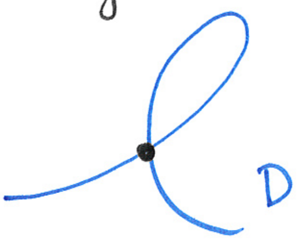
i) There is a λ s.t. $(x^*, y^*; \lambda)$ satisfy the Lagrange constraints FOC + C:

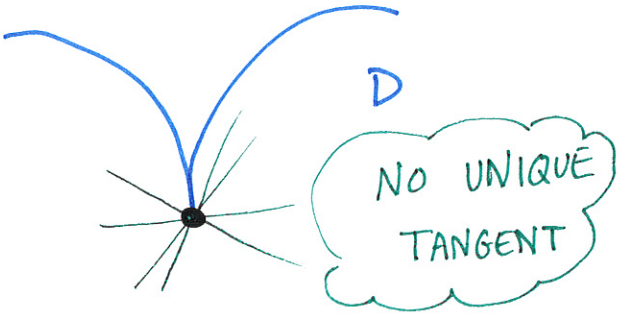
FOC: $\begin{cases} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{cases}$ and C: $g(x,y) = a$

OR ii) The constraint is degenerate at (x^*, y^*), i.e.,

$$g'_x = 0 \text{ and } g'_y = 0 \quad \text{and} \quad g(x,y) = a$$

Ex: In general, an extreme point with a degenerate constraint is a point where D doesn't have a unique tangent:





Ex ctd: $g(x,y) = 3x + 4y$

$g'_x = 3 \neq 0$

$g'_y = 4 \neq 0$, so case ii) of Theorem is not possible.