EBA1180 Mathematics for Data Science autumn 2023
Exercises

... if I couldn't formulate a problem in economic theory mathematically, I didn't know what I was doing.

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Lecture 13-14
Sec. 4.7, 7.9, 5.2-3, 4.9-10: Rational functions and asymptotes. Inverse functions. Exponential functions. Logarithms.

Here are recommended exercises from the textbook [SHSC].

Section **4.7** exercise 4
Section **7.9** exercise 1-5
Section **5.2** exercise 2a, 3, 4
Section **5.3** exercise 1, 3-5, 7, 9, 10
Section **4.9** exercise 1, 2, 4, 6
Section **4.10** exercise 1, 2, 6, 8-10

Problems for the exercise session
Wednesday 11 Oct. 12-17+ in D1-065

Problem 1 Determine the expression $f(x) = c + \frac{a}{x-b}$ of the hyperbolas (a-d) in figure 1.

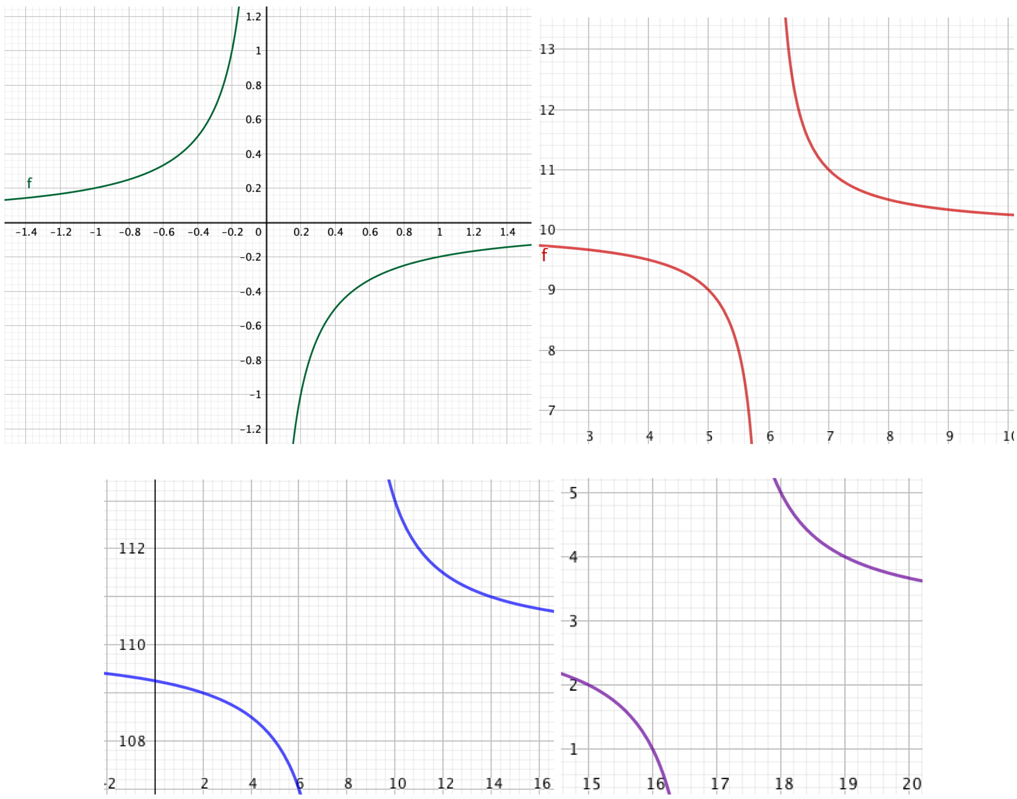

Figure 1: Hyperbolas a-d

Problem 2 Determine the asymptotes of the hyperbolas (a-d) in Problem 1.

Problem 3 Determine the asymptotes of the rational functions.

a) $f(x) = \frac{4x-10}{x-3}$ b) $f(x) = \frac{70-40x}{3-2x}$ c) $f(x) = \frac{12}{x^2+3}$

d) $f(x) = \frac{4x^2-28x+40}{x^2-4x+3}$ e) $f(x) = \frac{x^2+3x+5}{x-7}$ f) $f(x) = \frac{x^3-8}{x^2-10x+16}$

Problem 4 Suppose $g(x)$ is the inverse function of $f(x)$. Determine:

a) $g(10)$ if $f(3) = 10$ b) $f(g(5))$ c) $f(\sqrt{2})$ if $g(3) = \sqrt{2}$ d) $g(f(9))$

Problem 5 Determine the inverse function $g(x)$ and the domain D_g of the function $f(x)$ with domain D_f.

a) $f(x) = 2x - 3$ with $D_f =$ all numbers b) $f(x) = 0.5x + 1.5$ with $D_f =$ all numbers c) $f(x) = x^2 + 6x$ with $D_f = \langle \leftarrow , -3]$

d) $f(x) = 20 + \frac{1}{x-3}$ with $D_f = \langle 3, \rightarrow \rangle$ e) $f(x) = (x-1)^3 + 50$ with $D_f = [1, \rightarrow \rangle$

f) $f(x) = \begin{cases} \frac{10}{x} & \text{if } 0 < x \leqslant 10 \\ 2 - \frac{x}{10} & \text{if } 10 < x \leqslant 20 \end{cases}$

Problem 6 We have (approximately) $\ln 2 = 0.6931$ and $\ln 3 = 1.0986$ and $\ln 5 = 1.6094$. Use these numbers to determine the values (approximately) without using the ln-button on the calculator.

a) $\ln 250$ b) $\ln 625$ c) $\ln \frac{625}{216}$

d) $\ln \frac{1\,000\,000}{27}$ e) $\ln 130 - \ln 78$ f) $\ln \sqrt[10]{6}$

Problem 7 Solve the equations.

a) $e^x = 5$ b) $e^{2x+1} = 5$ c) $e^{2x+1} = 3e^{x+2}$

d) $\ln(x) = -2$ e) $\ln(7x-3) = -2$ f) $\ln(x-3) = \ln(2x+1) + 1$

g) $e^{2x} - 4e^x - 5 = 0$ h) $\frac{20 \ln \sqrt{x}}{1 - \ln x} = 10$

Problem 8 Solve the inequalities.

a) $e^x \geqslant 5$ b) $e^{2x+1} \geqslant 5$ c) $\ln(x) < -2$ d) $\ln(x-3) < -2$

e) $\frac{3e^x}{e^x+1} < 5$ f) $\ln \frac{3x-2}{x-7} \geqslant 0$

Problem 9 Determine the asymptotes of the function.

a) $f(x) = e^{-0.1x} + 23$ b) $f(x) = e^{x(10-x)} + 50$ c) $f(x) = \frac{100e^{0.04x}}{e^{0.04x}+50}$

d) $f(x) = \ln(10-x)$ e) $f(x) = \ln(x^2 - 400)$

f) $f(x) = \ln(120x + 10) - \ln(20x - 30)$, $D_f = \langle \frac{3}{2}, \rightarrow \rangle$

Problem 10 Determine the inverse function $g(x)$ and the domain D_g of the function $f(x)$ with domain D_f.

a) $f(x) = e^{\frac{x}{3}} - 1$ with $D_f = [0, \rightarrow \rangle$ b) $f(x) = 4\ln(x-10)$ with $D_f = [11, \rightarrow \rangle$

c) $f(x) = e^{\frac{2}{x+10}}$ with $D_f = [0, \rightarrow \rangle$ d) $f(x) = \ln(x^2 - 6x + 7)$ with $D_f = [0,1\rangle$

Answers

Problem 1
a) $f(x) = -\frac{1}{5x}$ b) $f(x) = 10 + \frac{1}{x-6}$ c) $f(x) = 110 + \frac{6}{x-8}$ d) $f(x) = 3 + \frac{2}{x-17}$

Problem 2
a) vertical asymptote: $x = 0$, horizontal asymptote: $y = 0$
b) vertical asymptote: $x = 6$, horizontal asymptote: $y = 10$
c) vertical asymptote: $x = 8$, horizontal asymptote: $y = 110$
d) vertical asymptote: $x = 17$, horizontal asymptote: $y = 3$

Problem 3
a) $f(x) = 4 + \frac{2}{x-3}$ so vertical asymptote: $x = 3$, horizontal asymptote: $y = 4$
b) $f(x) = 20 - \frac{10}{2x-3}$ so vertical asymptote: $x = \frac{3}{2}$, horizontal asymptote: $y = 20$
c) Since $x^2 + 3$ is positive for all x, $f(x)$ is defined for all x, so no vertical asymptote. Horizontal asymptote: $y = 0$
d) $f(x) = 4 - \frac{4(3x-7)}{(x-1)(x-3)}$ so vertical asymptotes: $x = 1$ and $x = 3$, horizontal asymptote: $y = 4$
e) $f(x) = x + 10 + \frac{75}{x-7}$ so vertical asymptote: $x = 7$, non-vertical asymptote: $y = x + 10$
f) $f(x) = x + 10 + \frac{84}{x-8}$ so vertical asymptote: $x = 8$, non-vertical asymptote: $y = x + 10$

Problem 4 a) $g(10) = 3$ b) $f(g(5)) = 5$ c) $f(\sqrt{2}) = 3$ d) $g(f(9)) = 9$

Problem 5

a) $g(x) = 0.5x + 1.5$ with D_g = all numbers

b) $g(x) = 2x - 3$, D_g = all numbers

c) $g(x) = -3 - \sqrt{x+9}$, $D_g = R_f = [-9, \rightarrow\rangle$

d) $g(x) = 3 + \frac{1}{x-20}$, $D_g = \langle 20, \rightarrow\rangle$

e) $g(x) = \sqrt[3]{x-50} + 1$, $D_g = [50, \rightarrow\rangle$

f)
$$g(x) = \begin{cases} \frac{10}{x} & \text{if } x \geqslant 1 \\ 20 - 10x & \text{if } 0 \leqslant x < 1 \end{cases}$$

Problem 6

a) $\ln 250 = \ln 2 + 3\ln 5 = 0.6931 + 3 \cdot 1.6094 = 5.5213$

b) $\ln 625 = 4\ln 5 = 4 \cdot 1.6094 = 6.4376$

c) $\ln \frac{625}{216} = 4\ln 5 - 3(\ln 3 + \ln 2) = 4 \cdot 1.6094 - 3(1.0986 + 0.6931) = 1.0625$

d) $\ln \frac{1\,000\,000}{27} = 6(\ln 5 + \ln 2) - 3\ln 3 = 6 \cdot (1.6094 + 0.6931) - 3 \cdot 1.0986 = 10.5192$

e) $\ln 130 - \ln 78 = \ln 5 + \ln 26 - \ln 3 - \ln 26 = 1.6094 - 1.0986 = 0.5108$

f) $\ln 6^{\frac{1}{10}} = \frac{1}{10} \cdot \ln 6 = \frac{1.0986+0.6931}{10} = 0.1792$

Problem 7

a) $x = \ln 5$ b) $x = \frac{1}{2}(\ln(5) - 1)$ c) $x = 1 + \ln(3)$

d) $x = e^{-2}$ e) $x = \frac{e^{-2}+3}{7}$ f) $x = -\dfrac{e+3}{2e-1}$

g) $x = \ln 5$ h) $x = e^{0,5}$

Problem 8

a) Because $\ln x$ is a strictly increasing function for $x > 0$ we can insert the left hand side and the right hand side into $\ln x$ and keep the inequality. It gives $x \geqslant \ln 5$.

b) We insert the left hand side and the right hand side into $\ln x$ and keep the inequality. It gives $x \geqslant \frac{1}{2}(\ln 5 - 1)$.

c) Because e^x is a strictly increasing function we can insert the left hand side and the right hand side into e^x and keep the inequality. It gives $0 < x < e^{-2}$.

d) We insert the left hand side and the right hand side into e^x and keep the inequality. It gives $3 < x < 3 + e^{-2}$.

e) All numbers on the number line (are called the real numbers and written as $\mathbb{R}$, i.e. $x \in \mathbb{R}$).

f) Note that the inequality only is defined for $x < \frac{2}{3}$ and for $x > 7$. We insert the left and right hand side into e^x and keep the inequality. This gives $\frac{3x-2}{x-7} \geqslant 1$ which we then solve: $x \leqslant -\frac{5}{2}$ or $x > 7$ (and this is within the domain of definition of the inequality). Alternate way of writing: $x \in \langle \leftarrow\, , -\frac{5}{2}] \cup \langle 7\, , \rightarrow \rangle$.

Problem 9

a) horizontal asymptote: $y = 23$ (when $x \to \infty$)

b) horizontal asymptote: $y = 50$ (when $x \to \pm\infty$)

c) horizontale asymptotes: $y = 100$ ($x \to \infty$) and $y = 0$ ($x \to -\infty$)

d) vertical asymptote: $x = 10$ ($y \to -\infty$ when $x \to 10^-$)

e) vertical asymptotes: $x = \pm 20$ ($y \to -\infty$ when $x \to -20^-$ and $y \to -\infty$ when $x \to 20^+$)

f) vertical asymptote: $x = \frac{3}{2}$, horizontal asymptote: $y = \ln 6$

Problem 10

a) $g(x) = 3\ln(x+1)$, $D_g = R_f = [0\, , \rightarrow\rangle$

b) $g(x) = e^{\frac{x}{4}} + 10$, $D_g = [0\, , \rightarrow\rangle$

c) $g(x) = \frac{2}{\ln x} - 10$, $D_g = \langle 1, \sqrt[5]{e}]$

d) $g(x) = 3 - \sqrt{e^x + 2}$, $D_g = \langle \ln 2, \ln 7]$