Exercise session problems

Problem 1.
Consider the linear system with unknowns x,y,z,w and extended matrix

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 2 & 1 & 1 & 0 & 2 \\ 4 & 2 & 1 & 2 & 0 \end{array}\right)$$

a) Use Gaussian elimination to transform the extended matrix to echelon form and to a reduced echelon form.

b) Determine which variables are dependent and which are free. How many degrees of freedom are there?

c) Write the solutions of the system in parameter form and determine all solutions with $z = 0$.

Problem 2.
Compute the following inner products when

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{v}_4 = \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix}$$

a) $\vec{v}_1 \cdot \vec{v}_2$
b) $\vec{v}_1 \cdot \vec{v}_3$
c) $\vec{v}_2 \cdot \vec{v}_2$
d) $(\vec{v}_1 - \vec{v}_2) \cdot \vec{v}_4$
e) $\vec{v}_1 \cdot (\vec{v}_2 + \vec{v}_3)$
f) $\vec{v}_1 \cdot (\vec{v}_2 - \vec{v}_3)$
g) $(\vec{v}_4 - \vec{v}_1) \cdot (\vec{v}_2 + \vec{v}_3)$
h) $(\vec{v}_1 - \vec{v}_2) \cdot (\vec{v}_1 - \vec{v}_2)$

Problem 3.
Find all the vectors that are orthogonal to the vector $\mathbf{v}$:

a) $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$
b) $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
c) $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$
d) $\mathbf{v} = \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix}$

Problem 4.
Determine $||\vec{v} - \vec{w}||$ when the vectors $\vec{v}, \vec{w}$ are orthogonal and have length $||\vec{v}|| = 3$ and $||\vec{w}|| = 4$.

Problem 5.
Find the natural domain D_f and range V_f of the function f:

a) $f(x,y) = 2x + 3y$
b) $f(x,y) = \sqrt{x + 3y}$
c) $f(x,y) = (2x - y)^{-3/2}$
d) $f(x,y) = 17x^{1.2}y^{3.4}$

Problem 6.
Consider the level curve $f(x,y) = c$ of a function $f(x,y)$. Draw the level curves for the given values of c in the same coordinate system and determine what kind of curve we get when we let c be an arbitrary value:

a) $f(x,y) = 12x - 3y$ og $c = -3, 0, 3$
b) $f(x,y) = xy$ og $c = -1, 0, 1$
c) $f(x,y) = x^2 + 2x + y^2 - 4y$ og $c = -9, -5, -1$
d) $f(x,y) = x^2 - 2x + 4y^2$ og $c = -2, -1, 0, 1$

Problem 7.
Use the level curves $f(x,y) = c$ from Problem 5 to determine whether the following functions have maximum- or minimum values:

a) $f(x,y) = 12x - 3y$

b) $f(x,y) = xy$

c) $f(x,y) = x^2 + 2x + y^2 - 4y$

d) $f(x,y) = x^2 - 2x + 4y^2$

Problem 8.
Describe the graph of $f(x,y) = 3x - 4y + 1$ geometrically. What is meant by a geometrical description is for example: *The graph of $f(x) = 3 - 2x$ is a straight line with slope -2 which intersects the y-axis in $y = 3$*, i.e., a precise geometrical description without using equations etc.

Problem 9.
Find the partial derivatives f'_x and f'_y when

a) $f(x,y) = 2x + 3y$

b) $f(x,y) = x^2 + y^2$

c) $f(x,y) = 4x^2 - 6xy + 9y^2$

d) $f(x,y) = x^2 - 2x + 4y^2$

e) $f(x,y) = x^3 - 3xy + y^3$

f) $f(x,y) = y^2 - x^3 + 3x$

g) $f(x,y) = x^2y^2 - x^2 - y^2 + 3$

h) $f(x,y) = \sqrt{x^2 + y^2}$

Problem 10.
Find the Hessian matrix $H(f)$, and compute $H(f)(1,1)$:

a) $f(x,y) = 2x + 3y$

b) $f(x,y) = x^2 + y^2$

c) $f(x,y) = 4x^2 - 6xy + 9y^2$

d) $f(x,y) = x^2 - 2x + 4y^2$

e) $f(x,y) = x^3 - 3xy + y^3$

f) $f(x,y) = y^2 - x^3 + 3x$

g) $f(x,y) = x^2y^2 - x^2 - y^2 + 3$

h) $f(x,y) = \sqrt{x^2 + y^2}$

Problem 11.
Find the stationary points of f, and classify them:

a) $f(x,y) = 2x + 3y$

b) $f(x,y) = x^2 + y^2$

c) $f(x,y) = 4x^2 - 6xy + 9y^2$

d) $f(x,y) = x^2 - 2x + 4y^2$

e) $f(x,y) = x^3 - 3xy + y^3$

f) $f(x,y) = y^2 - x^3 + 3x$

g) $f(x,y) = x^2y^2 - x^2 - y^2 + 3$

h) $f(x,y) = \sqrt{x^2 + y^2}$

Problem 12.
Find all stationary points and classify them:

a) $f(x,y) = xy(x^2 - y^2)$

b) $f(x,y) = x^2y + xy^3 + xy^2$

c) $f(x,y) = \sqrt{36 - 9x^2 - 4y^2}$

Oppgaver fra læreboken

Answers to the exercise session problems

Læreboken [E]:	Eriksen, *Matematikk for økonomi og finans*
Oppgaveboken [O]:	Eriksen, *Matematikk for økonomi og finans - Oppgaver og Løsningsforslag*

Oppgaver:	[E] 7.1.1 - 7.1.4, 7.2.1 - 7.2.2, 7.3.1 - 7.3.5, 7.4.1 - 7.4.2
Fullstendig løsning:	Se [O] Kap 7.1 - 7.4

Problem 1.

a)

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 0 & 3 & -5 & -8 & -20 \\ 0 & 0 & -1 & 2 & -4 \end{array}\right), \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & -2 & 4 \end{array}\right)$$

b) x,y,z dependent, w free, one degree of freedom and infinitely many solutions.

c) $(x,y,z,w) = (-4t-1, 6t, 2t+4, t)$, one solution $(7,-12,0,-2)$ with $z = 0$

Problem 2.

a) 0 b) 4 c) 2 d) 13

e) 4 f) -4 g) 18 h) 5

Problem 3.

All linear combinations of the following vectors:

a) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ b) $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ c) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ d) $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -7 \\ 4 \\ 0 \end{pmatrix}$

Problem 4.

5

Problem 5.

a) $D_f = \mathbb{R}^2,\ V_f = \mathbb{R}$

b) $D_f = \{(x,y) \in \mathbb{R}^2 : x + 3y \geq 0\},\ V_f = [0,\infty)$

c) $D_f = \{(x,y) \in \mathbb{R}^2 : 2x - y > 0\},\ V_f = (0,\infty)$

d) $D_f = \mathbb{R}^2,\ V_f = \mathbb{R}$

Problem 6.

a) Straight line with slope 4 which intersects the y-axis in $y = -c/3$

b) Hyperbola $y = c/x$ if $c \neq 0$, and the two axis if $c = 0$

c) Circle with radius $\sqrt{c+5}$ and center $(-1,2)$ if $c > -5$, one point $(-1,2)$ if $c = -5$, no points otherwise.

d) Ellipse with center in $(1,0)$ with half axis $a = \sqrt{c+1}$ and $b = \sqrt{c+1}/2$ when $c > -1$, one point $(1,0)$ if $c = -1$, and no points otherwise.

Problem 7.

a) Neither maximum nor minimum.

b) Neither maximum nor minimum.

c) No maximum, but the minimum value is $f_{min} = -5$.

d) No maximum, but the minimum value is $f_{min} = -1$

Problem 8.

The graph is the plane which intersects the z-axis in $z = 1$ and has the normal vector $(3, -4, -1)$.

Problem 9.

a) $f'_x = 2,\ f'_y = 3$

b) $f'_x = 2x,\ f'_y = 2y$

c) $f'_x = 8x - 6y,\ f'_y = -6x + 18y$

d) $f'_x = 2x - 2,\ f'_y = 8y$

e) $f'_x = 3x^2 - 3y,\ f'_y = -3x + 3y^2$

f) $f'_x = -3x^2 + 3,\ f'_y = 2y$

g) $f'_x = 2x(y^2 - 1),\ f'_y = 2y(x^2 - 1)$

h) $f'_x = \dfrac{x}{\sqrt{x^2 + y^2}},\ f'_y = \dfrac{y}{\sqrt{x^2 + y^2}}$

Problem 10.

a) $H(f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

b) $H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

c) $H(f) = \begin{pmatrix} 8 & -6 \\ -6 & 18 \end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix} 8 & -6 \\ -6 & 18 \end{pmatrix}$

d) $H(f) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix}$

e) $H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$

f) $H(f) = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$

g) $H(f) = \begin{pmatrix} 2(y^2 - 1) & 4xy \\ 4xy & 2(x^2 - 1) \end{pmatrix},\ H(f)(1{,}1) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$

h) $H(f) = (x^2 + y^2)^{-3/2} \cdot \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix},\quad H(f)(1{,}1) = \begin{pmatrix} \sqrt{2}/4 & -\sqrt{2}/4 \\ -\sqrt{2}/4 & \sqrt{2}/4 \end{pmatrix}$

Problem 11.

a) None

b) (0,0) is a local min.

c) (0,0) is a local min min.

d) (1,0) is a local min min.

e) (0,0) is a saddle point and (1,1) is a local min.

f) (1,0) is a saddle point and $(-1{,}0)$ is a local min.

g) (0,0) is a local max and $(\pm 1, \pm 1)$ is a saddle point.

h) none; (0,0) is a critical point.

Problem 12.

a) (0,0) is a saddle point.

b) (0,0), $(0, -1)$ is a saddle point, $(3/25, -3/5)$ is a local max.

c) (0,0) is a local (and global) max.