Exercise session problems

Problem 1.
Consider the optimization problem

$$\max / \min f(x,y) = x^2 - 4x + y^2 - 2y + 6 \text{ when } 0 \le x \le 4,\ 0 \le y \le 3$$

a) Sketch the set D of admissible points in the xy-coordinate system.

b) Find any stationaty points for f in the interior of D, and mark these in the figure.

c) Describe the level curve $f(x,y) = c$ geometrically, and draw these level curves in the figure when $c = 1$, $c = 2$ and $c = 5$.

d) Use the figure to find maximum- and minimum points.

e) Find the maximum and minimum points by calculation, and compare to the points you found above.

Problem 2.
Consider the optimization problem

$$\max / \min f(x,y) = x^2 - 6x + y^2 - 4y + 13 \text{ when } x \ge 0,\ y \ge 0,\ 2x + 3y \le 6$$

a) Sketch the set D of admissible points in the xy-coordinate system.

b) Find any stationary points for f in the interior of D, and mark these in the figure.

c) Describe the level curve $f(x,y) = c$ geometrically, and draw these level curves in the figure when $c = 0$, $c = 1$ and $c = 4$.

d) Use the figure to find the maximum- and minimum points.

e) Find the maximum- and minimum points by calculation and compare with the points you found above.

Problem 3.
Find any maximum- and/or minimum points and maximum- and/or minimum values:

a) $\max / \min f(x,y) = x^2 + 4y^2$ when $3x - y = 37$

b) $\max / \min f(x,y) = x^2 + 4y^2$ when $xy = 8$

c) $\max / \min f(x,y) = x^2y^2 - x^2 - y^2 + 16$ when $xy = 4$

Problem 4.
Solve the optimization problem: $\max / \min \ f(x,y) = x^3 + 3xy + y^3 \quad \text{when} \quad xy = 1$

Problem 5.
Solve the optimization problem: $\max \ f(x,y) = (x - y)e^{2xy} \quad \text{when} \quad 0 \le x \le 1,\ 0 \le y \le 1$

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Problem 6.
Consider the level curve $g(x,y) = 0$, where g is the function $g(x,y) = x^3 + xy + y^2$.

a) Find all points on the level curve with $x = -2$, and determine the tangent in each of these points.
b) Find the maximum value of $f(x,y) = x$ under the constraint $x^3 + xy + y^2 = 0$.

Problem 7.
Consider the curve C with equation $y(x^2 + y^2) = 2(x^2 - y^2)$.

a) Find all points on the curve C where $y = -1$.
b) Find the tangent of C in each point where $y = -1$.
c) Solve the optimization problem: $\max / \min \; f(x,y) = y \quad \text{when} \quad y(x^2 + y^2) = 2(x^2 - y^2)$

Problem 8.
Consider the function defined by $f(x,y) = 1 + x^2 + y^2 + x^2y^2$.

a) Find all stationary points for f.
b) Calculate the Hessian matrix of f, and use this to classify the stationary points.
c) Determine whether f has global maximum- or minimum values.
d) Solve the Lagrange problem: $\max f(x,y) = x^2 + y^2 + x^2y^2$ when $x^2 + 2y^2 = 5$

Problem 9.
Consider the Lagrange problem $\max / \min f(x,y) = x + 2y - \sqrt{36 - x^2 - 4y^2} \quad \text{when} \quad x^2 + 4y^2 = 36$.

a) Find the points on the level curve $x^2 + 4y^2 = 36$ where the tangent has slope $y' = 1/2$.
b) Make a sketch of $D = \{(x,y) : x^2 + 4y^2 = 36\}$. Is D bounded? What kind of curve is this?
c) Solve the Lagrange problem and find the maximum- and minimum value.
d) Solve the new optimization problem we get when we change the constraint to $x^2 + 4y^2 \leq 36$.

Answers to the exercise session problems

Problem 1.
a) Rectangle in the first quadrant with sides 4 and 3.
b) (2,1)
c) Circles with center in (2,1) and radius $\sqrt{c-1}$ for $c > 1$.
d) Maximum: (0,3),(4,3) and minimum: (2,1)
e) Maximum: (0,3),(4,3) and minimum: (2,1)

Problem 2.

a) 90 degree triangle in the first quadrant with sides 3 and 2.

b) None.

c) Circles with center in (3,2) and radius $\sqrt{c}$ for $c > 0$.

d) Maximum: (0,0) and minimum: Ca (2,0.7)

e) Maximum: (0,0) and minimum: (27/13,8/13)

Problem 3.

a) $f_{\text{min}} = 148$ in the point $(x,y) = (12,-1)$, no maximum value.

b) $f_{\text{min}} = 32$ in the points $(x,y) = (4,2), (-4,-2)$, no maximum value.

c) $f_{\text{max}} = 24$ in the points $(x,y) = (2,2), (-2,-2)$, no minimum value.

Problem 4.

Neither maximum nor minimum exist.

Problem 5.

$f_{\text{max}} = e/2 \approx 1.359$.

Problem 6.

a) $y = -8x/3 - 4/3$ in $(-2,4)$ and $y = 5x/3 + 4/3$ in $(-2,-2)$

b) $f_{\text{max}} = 1/4$

Problem 7.

a) $(\pm\sqrt{1/3},-1)$

b) $y = 2 \mp 3\sqrt{3}x$

c) $f_{\text{min}} = -2$, no maximum value.

Problem 8.

a) (0,0)

b) local minimum point.

c) $f_{\text{min}} = 1$, no maximum value.

d) $f_{\text{max}} = 7$

Problem 9.

a) $(3\sqrt{2},-3\sqrt{2}/2), (-3\sqrt{2},3\sqrt{2}/2)$

b) Yes, ellipse med half axes $a = 6$ and $b = 3$ with center in (0,0)

c) $f_{\text{max}} = 6\sqrt{2}$, $f_{\text{min}} = -6\sqrt{2}$

d) $f_{\text{max}} = 6\sqrt{2}$, $f_{\text{min}} = -6\sqrt{3}$