

EBA1180 Mathematics for Data Science
autumn 2024
Exercises

... if I couldn't formulate a problem in economic theory mathematically, I didn't know what I was doing.

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Lecture 4 and 5

Sec. 11.4.1-3, 6.11.4, 11.2, 11.3.2, 11.4.4-10, 4.9.2:

Geometric series. Annuities. Euler's number and continuous compounding of interest.

Here are recommended exercises from the textbook [SHSC].

- Section 11.2 exercise 1-3, 5
- Section 4.9 exercise 1, 2, 4, 5
- Section 11.3 exercise 1b, 2b
- Section 11.4 exercise 1-3, 6, 7
- Section 11.5 exercise 6
- Section 11.6 exercise 3

Problems for the exercise session
Wednesday 4 Sept. from 12 in CU1-067

Problem 1 Calculate the sum of the series.

- a) $1 + 1.04 + 1.04^2 + 1.04^3 + \dots + 1.04^{10}$.
- b) $1 + 1.04 + 1.04^2 + 1.04^3 + \dots + 1.04^{20}$.
- c) $1 + 1.04 + 1.04^2 + 1.04^3 + \dots + 1.04^n$.
- d) $30\,000 \cdot 1.04^{20} + 30\,000 \cdot 1.04^{19} + 30\,000 \cdot 1.04^{18} + \dots + 30\,000 \cdot 1.04^2 + 30\,000 \cdot 1.04$.
- e) Describe a financial situation where the sum in (d) is used.
- f) $1 + \frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \dots + \frac{1}{1.04^{20}}$.
- g) Explain why 1.04^{20} multiplied with the sum in (f) gives the sum in (b).
- h) $1 + \frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \dots + \frac{1}{1.04^n}$.
- i) $\frac{30\,000}{1.04} + \frac{30\,000}{1.04^2} + \frac{30\,000}{1.04^3} + \dots + \frac{30\,000}{1.04^{20}}$.
- j) Describe a financial situation where the sum in (i) is used.

Problem 2 Suppose you are paid 500 000 every year for n years with the first payment in one year from now. Assume the interest is 3.5%.

- a) Write down the geometric series which gives the present value of the cash flow.
- b) Use the geometric series to compute the present value of the cash flow for $n = 10$, $n = 20$, $n = 40$, $n = 80$ and $n = 1000$.
- c) Compute the present value of the cash flow if it continues forever.

Problem 3 The nominal annual interest is 4.8%.

- a) Assume annual compounding. Determine the annual rate of change (growth factor). Determine the rate of change for 10 years. Determine the effective interest for 10 years.
- b) Assume quarterly compounding. Determine the annual rate of change and the effective interest. Determine the rate of change for 10 years. Determine the effective interest for 10 years.
- c) Assume monthly compounding. Determine the annual rate of change and the effective interest. Determine the rate of change for 10 years. Determine the effective interest for 10 years.

- d) Assume daily compounding. Determine the annual rate of change and the effective interest. Determine the rate of change for 10 years. Determine the effective interest for 10 years.
- e) Assume continuous compounding. Determine the annual rate of change and the effective interest. Determine the rate of change for 10 years. Determine the effective interest for 10 years.

Problem 4 You deposit 30 000 into an account with 2.9% nominal interest.

- a) Assume annual compounding.
- i) Compute the balance after 10 years.
 - ii) Determine the rate of change and the relative change for the 10 years.
- b) Assume continuous compounding.
- i) Compute the balance after 10 years.
 - ii) Determine the rate of change and the relative change for the 10 years.
 - iii) Determine the (annual) effective interest.

Problem 5 You consider an investment of 2 million in an asset which can be sold for 5 million after 20 years.

- a) With annual compounding, compute the internal rate of return.
- b) With quarterly compounding, compute the internal rate of return.
- c) With monthly compounding, compute the internal rate of return.
- d) With continuous compounding, compute the internal rate of return. (Hint: Try different rates.)

Problem 6 Hege considers a mortgage with 25 annual payments. She believes she will be able to pay 120 000 each year. First payment is one year from now.

- a) Assume the interest is 2.0% with annual compounding. Determine the geometric series which gives the present value of the cash flow. Use it to calculate how much Hege can borrow.
- b) Assume the interest is 2.0% with continuous compounding. Determine the geometric series which gives the present value of the cash flow. Use it to calculate how much Hege can borrow.
- c) Compare the answers in (a) and (b).

Problem 7 Suppose you will be paid 300 000 every year in n years with the first payment a year from now. Suppose the interest is 3.5% with continuous compounding.

- a) Write down the geometric series which gives the present value of the cash flow.
- b) Use the geometric series to compute the present value of the cash flow for $n = 10$, $n = 20$, $n = 40$, $n = 80$ og $n = 1000$.
- c) Use the geometric series to compute the present value of the cash flow if it continues forever.

Problem 8 Suppose a constant amount $A = 40\,000$ (the annuity) is paid every year in n years with the first payment one year from now. Suppose the nominal interest is r with continuous compounding.

- a) Write down the geometric series which gives the present value of the cash flow if $n = 25$ og $r = 2.6\%$. Use this series to calculate the present value.
- b) Assume the annuity is paid forever. Write down the infinite geometric series which gives the present value of the cash flow if $r = 2.6\%$. Use this series to calculate the present value.
- c) Assume the annuity is paid forever. Determine the interest r such that the present value (K_0) becomes 3 million. (Hint: Try different rates.)
- d) Explain why (c) gives the equation

$$e^r = \frac{K_0 + A}{K_0} = \frac{3\,000\,000 + 40\,000}{3\,000\,000} = 1.0133$$

Answers

Problem 1

- a) $\frac{1.04^{11}-1}{0.04} = 13.49$.
- b) $\frac{1.04^{21}-1}{0.04} = 31.97$.
- c) $\frac{1.04^{n+1}-1}{0.04}$.
- d) $30\,000 \cdot 1.04 \cdot \frac{1.04^{20}-1}{0.04} = 929\,076.05$.
- e) A deposit of 30 000 every year for 20 years (starting today) into an account with 4% interest and annual compounding will give the sum as future value after 20 years.
- f) We read the geometric series backwards: $\frac{1}{1.04^{20}} \cdot \frac{1.04^{21}-1}{0.04} = 14.59$.
- g) $(1 + \frac{1}{1.04} + \frac{1}{1.04^2} + \frac{1}{1.04^3} + \dots + \frac{1}{1.04^{20}}) \cdot 1.04^{20} = 1.04^{20} + 1.04^{19} + \dots + 1.04^2 + 1.04 + 1$.
- h) $\frac{1}{1.04^n} \cdot \frac{1.04^{n+1}-1}{0.04}$.
- i) $\frac{30\,000}{1.04^{20}} \cdot \frac{1.04^{20}-1}{0.04} = 407\,709.79$.
- j) The sum represents the present value (what you can borrow) for a 30 000 annuity (starting a year from now) with 4% interest and yearly compounding running for 20 years.

Problem 2

- a) $\frac{500\,000}{1.035} + \frac{500\,000}{1.035^2} + \frac{500\,000}{1.035^3} + \dots + \frac{500\,000}{1.035^n}$.
- b) $n = 10: \frac{500\,000}{1.035^{10}} \cdot \frac{1.035^{10}-1}{0.035} = 4\,158\,302.66$, $n = 20: \frac{500\,000}{1.035^{20}} \cdot \frac{1.035^{20}-1}{0.035} = 7\,106\,201.65$,
 $n = 40: \frac{500\,000}{1.035^{40}} \cdot \frac{1.035^{40}-1}{0.035} = 10\,677\,536.17$, $n = 80: \frac{500\,000}{1.035^{80}} \cdot \frac{1.035^{80}-1}{0.035} = 13\,374\,387.83$ and
 $n = 1000: \frac{500\,000}{1.035^{1000}} \cdot \frac{1.035^{1000}-1}{0.035} = 14\,285\,714.29$.
- c) $\frac{500\,000}{1.035^n} \cdot \frac{1.035^n-1}{0.035} = 500\,000 \cdot \frac{1-\frac{1}{1.035^n}}{0.035}$ which approaches $500\,000 \cdot \frac{1}{0.035} = 14\,285\,714.29$ more and more as n becomes bigger and bigger (" n approaches infinity", often written " $n \rightarrow \infty$ ").

Problem 3

- a) Annual rate of change: 1.048, rate of change for 10 years: $1.048^{10} = 1.5981$, effective interest for 10 years: 59.81%.
- b) Annual rate of change: 1.0489, rate of change for 10 years: 1.6115, effective interest for 10 years: 61.15%.
- c) Annual rate of change: 1.0491, rate of change for 10 years: 1.6145, effective interest for 10 years: 61.45%.
- d) Annual rate of change: 1.0492, rate of change for 10 years: 1.6160, effective interest for 10 years: 61.60%.
- e) Annual rate of change: 1.0492, rate of change for 10 years: 1.6161, effective interest for 10 years: 61.61%.

Problem 4

- a) i) 39 927.77
 ii) rate of change: 1.3309, relative change: 33.09%
- b) i) 40 092.82
 ii) rate of change: 1.3364, relative change: 33.64%
 iii) 2.94%

Problem 5

- a) $2.5^{\frac{1}{20}} - 1 = 4.69\%$
- b) 4.61%
- c) 4.59%
- d) Obtain the equation $e^r = 2.5^{\frac{1}{20}} = 1.0469$ and try: $r = 4.58\%$.

Problem 6

- a) Present value:
 $120\,000 \cdot \frac{1}{1.02} + 120\,000 \cdot \frac{1}{1.02^2} + 120\,000 \cdot \frac{1}{1.02^3} + \dots + 120\,000 \cdot \frac{1}{1.02^{24}} + 120\,000 \cdot \frac{1}{1.02^{25}}$.
 Mortgage: 2 342 814.78

b) Present value:

$$120\,000 \cdot \frac{1}{e^{0.02}} + 120\,000 \cdot \frac{1}{(e^{0.02})^2} + 120\,000 \cdot \frac{1}{(e^{0.02})^3} + \dots + 120\,000 \cdot \frac{1}{(e^{0.02})^{24}} + 120\,000 \cdot \frac{1}{(e^{0.02})^{25}}.$$

$$\text{Mortgage: } 120\,000 \cdot \frac{1}{e^{0.02 \cdot 25}} \cdot \frac{e^{0.02 \cdot 25} - 1}{e^{0.02} - 1} = 2\,337\,286.57$$

c) With continuous compounding Hege can borrow slightly less since then the effective interest she has to pay is slightly higher.

Problem 7

a)

$$300\,000 \cdot \frac{1}{e^{0.035}} + 300\,000 \cdot \frac{1}{(e^{0.035})^2} + \dots + 300\,000 \cdot \frac{1}{(e^{0.035})^{n-1}} + 300\,000 \cdot \frac{1}{(e^{0.035})^n}$$

b) The sum of the geometric series: $300\,000 \cdot \frac{1}{(e^{0.035})^n} \cdot \frac{(e^{0.035})^n - 1}{e^{0.035} - 1}$. For $n = 10$: 2 487 206.55 for $n = 20$: 4 239 911.38 for $n = 40$: 6 345 389.07 for $n = 80$: 7 910 142.75 for $n = 1000$: 8 422 303.55.

c) $300\,000 \cdot \frac{1}{(e^{0.035})^n} \cdot \frac{(e^{0.035})^n - 1}{e^{0.035} - 1} = 300\,000 \cdot \frac{1 - (e^{0.035})^{-n}}{e^{0.035} - 1}$ approaches $300\,000 \cdot \frac{1}{e^{0.035} - 1} = 8\,422\,303.55$ when n grows bigger and bigger.

Problem 8

a)

$$\begin{aligned} & 40\,000 \cdot \frac{1}{e^{0.026}} + 40\,000 \cdot \frac{1}{(e^{0.026})^2} + \dots + 40\,000 \cdot \frac{1}{(e^{0.026})^{24}} + 40\,000 \cdot \frac{1}{(e^{0.026})^{25}} \\ &= 40\,000 \cdot \frac{1}{(e^{0.026})^{25}} \cdot \frac{(e^{0.026})^{25} - 1}{e^{0.026} - 1} = 40\,000 \cdot \frac{1}{e^{0.026 \cdot 25}} \cdot \frac{e^{0.026 \cdot 25} - 1}{e^{0.026} - 1} = 725\,796.53 \end{aligned}$$

b)

$$\begin{aligned} & 40\,000 \cdot \frac{1}{e^{0.026}} + 40\,000 \cdot \frac{1}{(e^{0.026})^2} + \dots + 40\,000 \cdot \frac{1}{(e^{0.026})^n} + \dots \\ &= 40\,000 \cdot \frac{1}{e^{0.026} - 1} = 1\,518\,548.20 \end{aligned}$$

c) Obtain the equation $e^r = 1.0133$ and try: $r = 1.32\%$.