Course paper 1 - EBA1180[1] Mathematics for Data Science
14 – 21 October 2022

SOLUTIONS

Problem 1

a) This is a geometric series with first term $a_1 = 25\,000 \cdot 1.004^{38}$, the number of terms $n = (276-36) : 2 = 120$ and the multiplication factor $k = 1.004^2$. Then the sum is

$$25\,000 \cdot 1.004^{38} \cdot \frac{1.004^{240}-1}{1.004^2-1} = \underline{\underline{5\,831\,740.85}}$$

b) The sum may be the balance (future value) 23 years from now if 4.8% is the nominell interest with monthly compounding (with monthly growth factor $1 + \frac{4.8\%}{12} = 1.004$), the deposit is 25 000 every second month with the first deposit today and the last 19 years and 10 months from now (that is 120 deposits) which then is left for 3 years and 2 months after the last deposit.

Problem 2

a) Annual growth factor is e^r. The present value of the cash flow equals the sum of the present values of the payments, that is

$$\underline{\underline{\frac{A}{e^{4r}} + \frac{B}{e^{6r}} + \frac{C}{e^{9r}}}}$$

If $A = 20$, $B = 30$, $C = 50$ og $r = 10\%$ then $\frac{20}{e^{0.4}} + \frac{30}{e^{0.6}} + \frac{50}{e^{0.9}} = \underline{\underline{50.20}}$

b) The future value of the cash flow after 7 years is the sum of the future values of the payments, that is

$$\underline{\underline{Ae^{3r} + Be^{r} + \frac{C}{e^{2r}}}}$$

which also is the future value 7 years from now of the present value of the cash flow. Inserting the values we hence get $50.20 \cdot e^{0.7} = \underline{\underline{101.09}}$.

c) i) We get the equation

$$\frac{20}{e^{4r}} = \frac{30}{e^{6r}} \quad \text{that is} \quad \frac{e^{6r}}{e^{4r}} = \frac{30}{20} \quad \text{that is} \quad e^{2r} = \frac{3}{2}$$

Insert the left and the right hand side into ln(−) and get $2r = \ln 3 - \ln 2$, that is $\underline{\underline{r = (\ln 3 - \ln 2) : 2 = 20.27\%}}$.

ii) We get the inequality

$$\frac{50}{e^{9r}} > \frac{30}{e^{6r}} \quad \text{that is} \quad \frac{50}{30} > \frac{e^{9r}}{e^{6r}} \quad \text{that is} \quad e^{3r} < \frac{5}{3}$$

Insert the left and the right hand side into ln(−). Because $\ln(x)$ is a strictly increasing function the new inequality is equivalent: $3r < \ln 5 - \ln 3$, that is $\underline{\underline{r < (\ln 5 - \ln 3) : 3 = 17.03\%}}$

d) We get the inequalities

i) $\frac{30}{e^{6r}} > \frac{20}{e^{4r}}$, that is $e^{2r} < 3/2$, that is $r < (\ln 3 - \ln 2)/2 = 20.27\%$

ii) $\frac{30}{e^{6r}} > \frac{50}{e^{9r}}$, that is $e^{3r} > 5/3$, that is $r > (\ln 5 - \ln 3)/3 = 17.03\%$

iii) $\frac{20}{e^{4r}} < \frac{50}{e^{9r}}$, that is $e^{5r} < 5/2$, that is $r < (\ln 5 - \ln 2)/5 = 18.33\%$

Hence $\underline{\underline{17.03\% < r < 18.33\%}}$.

[1] Exam code EBA11801

Problem 3

a) We have $(x-(-2+\sqrt{7}))(x-(-2-\sqrt{7})) = x^2+4x-3$. Hence the second degree polynomial equation $x^2+4x-3=0$ has the given roots.

b) We have $(x-\sqrt{10})(x+\sqrt{10}) = x^2-10$ and $(x-k)(x^2-10) = x^3-kx^2-10x+10k$. Hence the third degree polynomial equation $x^3-kx^2-10x+10k=0$ has the given roots.

Problem 4

a) We square both sides of the equation and get $4x+9 = x^2+2x+1$, that is $x^2-2x-8=0$ which gives $x=-2$ or $x=4$.

$x=-2$: The left hand side becomes $= \sqrt{4\cdot(-2)+9} = 1$ while the right hand side is $-2+1=-1$, hence no solution.

$x=4$: The left hand side becomes $= \sqrt{4\cdot 4+9} = 5$ and the right hand side is also $4+1=5$, hence a solution.

Conclusion: $x=4$ is the only solution.

b) We do the the same thing with this equation and get $x^2+2(t-2)x+t^2-9=0$. This second degree polynomial equation has no solutions if and only if $[2(t-2)]^2-4\cdot 1\cdot(t^2-9)<0$, that is $4t^2-16t+16-4t^2+36<0$, that is $t>3.25$. Then the original equation cannot have any solutions either.

c) Put $u=x^2$ and get the second degree polynomial equation $u^2+6u=k$, that is $(u+3)^2=k+9$ which has solutions $u=-3\pm\sqrt{k+9}$ exactly if $k\geqslant -9$. Substituting back and get $x^2=-3\pm\sqrt{k+9}$. This equation has has solutions exactly if $-3\pm\sqrt{k+9}\geqslant 0$ (and $k\geqslant -9$), that is $\pm\sqrt{k+9}\geqslant 3$. The only possibility is $\sqrt{k+9}\geqslant 3$, that is $k\geqslant 0$.

Problem 5

a) Since we have 0 on the right hand side and factorised left hand side we can use a sign diagram:

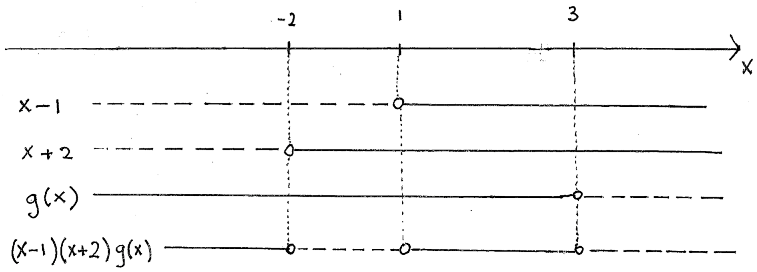

Figure 1: Sign diagram

We get $x\leqslant -2$ or $1\leqslant x\leqslant 3$ which also can be written as $x\in\langle\leftarrow, -2]$ or $x\in[1,3]$.

b) Note that the inequality only is defined for $x>0$. Again we have 0 on the right hand side and factorised left hand side we can use a sign diagram (e^{2-x} is greater than 0 for all x):

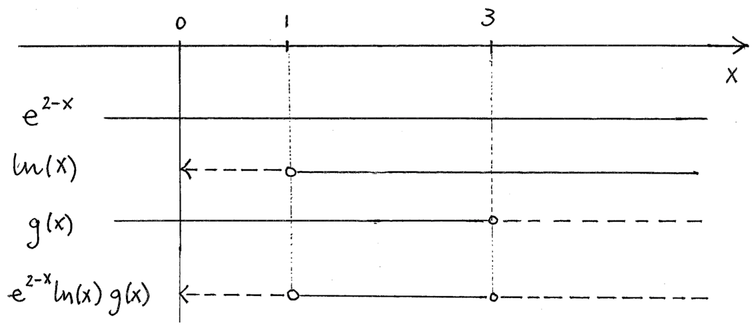

Figure 2: Sign diagram

It gives $0 < x < 1$ or $x > 3$ which also can be written as: $x \in \langle 0, 1\rangle$ or $x \in \langle 3, \rightarrow\rangle$.

c) Here we first have to collect the terms on the left hand side and make a common fraction:

$$\frac{(3x-5)g(x)}{x-5} - g(x) \leqslant 0 \quad \text{that is} \quad \frac{(3x-5)g(x)-(x-5)g(x)}{x-5} \leqslant 0 \quad \text{that is} \quad \frac{2x \cdot g(x)}{x-5} \leqslant 0$$

Then we can use a sign diagram:

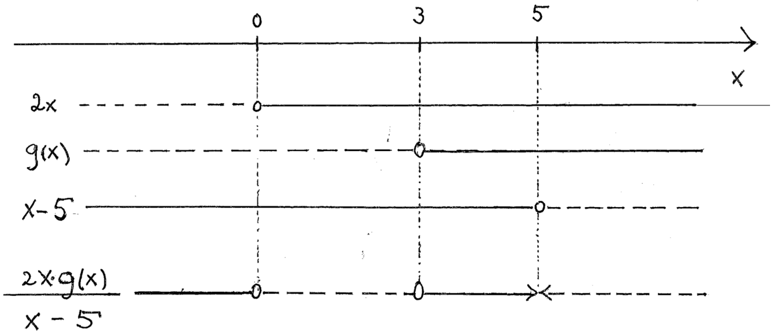

Figure 3: Sign diagram

We get $0 \leqslant x \leqslant 3$ or $x > 5$ which also can be written as: $x \in [0, 3]$ or $x \in \langle 5, \rightarrow\rangle$.

d) The inequality is on standard form and the numerator is always positive (at least 4). The denominator, and hence the fraction, is positive for $x > \ln(10)/2$ which also can be written as $x \in \langle \ln(10)/2, \rightarrow\rangle$.

Problem 6

a) We perform the polynomial division $f(x) : g(x)$ the usual way and get

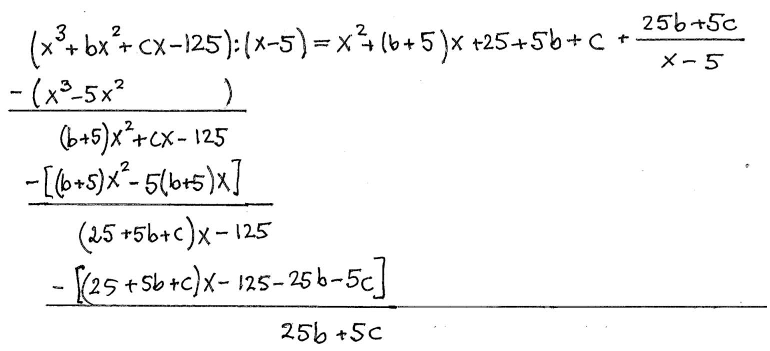

Figure 4: Polynomial division

$$\text{Hence} \quad (x^3 + bx^2 + cx - 125) : (x-5) = x^2 + (b+5)x + 25 + 5b + c + \frac{25b+5c}{x-5}$$

b) The remainder in the polynomial division equals $25b + 5c$ and $g(x)$ is a factor in $f(x)$ precisely when the remainder equals 0, that is $5(5b + c) = 0$, that is precisely when $5b + c = 0$.

Problem 7

a) Since $f(75) = f(105)$ the symmetry axis is $x = 90$ which gives $s = 90$ in the standard form $f(x) = a(x-s)^2 + d$ and the minimal value gives $d = -3.5$, that is $f(x) = a(x-90)^2 - 3.5$. From $f(105) = 4$ we get the equation $a(105-90)^2 - 3.5 = 4$, that is $225a = 7.5$, that is $a = \frac{1}{30}$ and $f(x) = \frac{1}{30}(x-90)^2 - 3.5$.

b) In the standard form $g(x) = c + \frac{a}{x-b}$ we have been given $b = 0$ and $c = 100$, that is $g(x) = 100 + \frac{a}{x}$. From $g(5) = 98$ we hence get the equation $100 + \frac{a}{5} = 98$ which gives $a = -10$, that is $\underline{\underline{g(x) = 100 - \frac{10}{x}}}$.

Problem 8

a) The four points are the corners of a square with side length 4 and centre $(8, 6)$ which also has to be the centre of the the circle. The radius of the circle then is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. Hence the circle equation on standard form is $\underline{\underline{(x-8)^2 + (y-6)^2 = 8}}$.

b) The centre of the ellipse is also $(8, 6)$ which gives the standard form

$$\frac{(x-8)^2}{a^2} + \frac{(y-6)^2}{b^2} = 1 \qquad (*)$$

Because C is supposed to be a point on the ellipse

$$\frac{(10-8)^2}{a^2} + \frac{(8-6)^2}{b^2} = 1 \quad \text{that is} \quad \frac{4}{a^2} + \frac{4}{b^2} = 1 \qquad (**)$$

The condition on the semi-axes is $a > b$. If we try with $a = 3$ we get $\frac{4}{9} + \frac{4}{b^2} = 1$ which we solve and get $b = \frac{6}{\sqrt{5}}$ which is less than 3. Hence $\underline{\underline{\frac{(x-8)^2}{9} + \frac{(y-6)^2}{7.2} = 1}}$ is an example of such an ellipse equation:

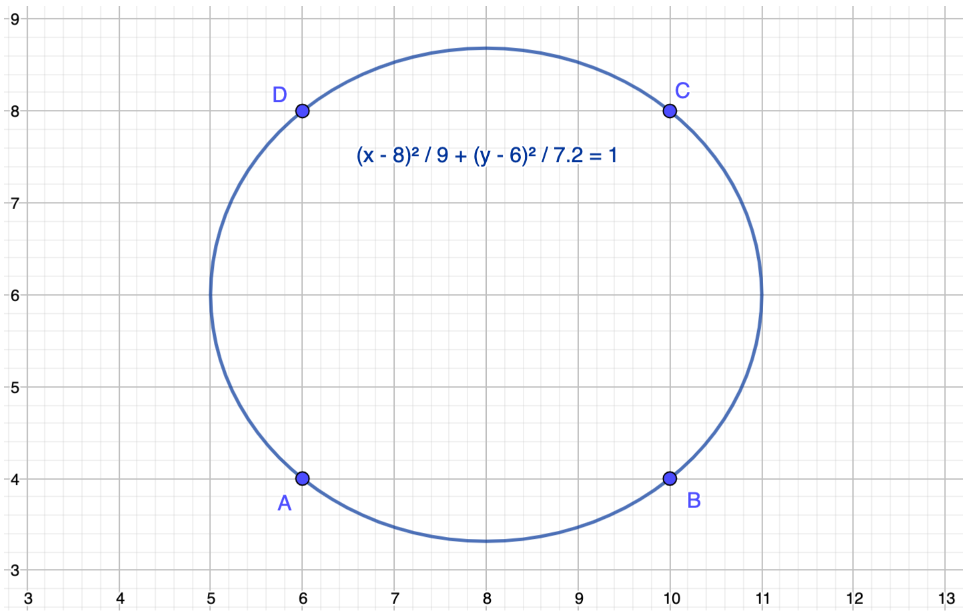

Figure 5: Ellipse

c) If the origin $(0, 0)$ is supposed to be a point on the ellipse, $x = 0 = y$ should satisfy the ellipse equation $(*)$, that is

$$\frac{(0-8)^2}{a^2} + \frac{(0-6)^2}{b^2} = 1 \quad \text{that is} \quad \frac{64}{a^2} + \frac{36}{b^2} = 1 \qquad (***)$$

Then $(**)$ and $(***)$ are two equations for a and for b. But they don't have common solutions, which can be shown in several ways. E.g. if we multiply both sides of $(**)$ with 9 we get $\frac{36}{a^2} + \frac{36}{b^2} = 9$. If we subtract the left hand side of this equation from the left hand side of $(***)$ and correspondingly on the right hand side we get the equation $\frac{28}{a^2} = -8$ which doesn't have any solutions since the left hand side always is positive. Then there is no ellipse as in $(*)$ which passes through the origin.

Problem 9

a) We put $y = 10 + \frac{0.2}{x-3}$ and solve for x. We subtract 10 on each side and get $y - 10 = \frac{0.2}{x-3}$. Multiplication by $x-3$ on each side gives $(x-3)(y-10) = 0.2$ and division with $y - 10$ on each side gives $x - 3 = \frac{0.2}{y-10}$, that is $x = 3 + \frac{0.2}{y-10}$. Switching variables gives $\underline{\underline{g(x) = 3 + \frac{0.2}{x-10}}}$. In general $D_g = V_f$. We see that $f(x)$ has a vertical asymptote $x = 3$ and that $f(x)$ grows without bounds when x approaches 3 from above. On the other hand $f(x)$ approaches 10 from above when x increases without bounds. This gives $D_g = V_f = \underline{\underline{\langle 10, \infty \rangle}}$. Moreover $V_g = D_f = \underline{\underline{\langle 3, \infty \rangle}}$.

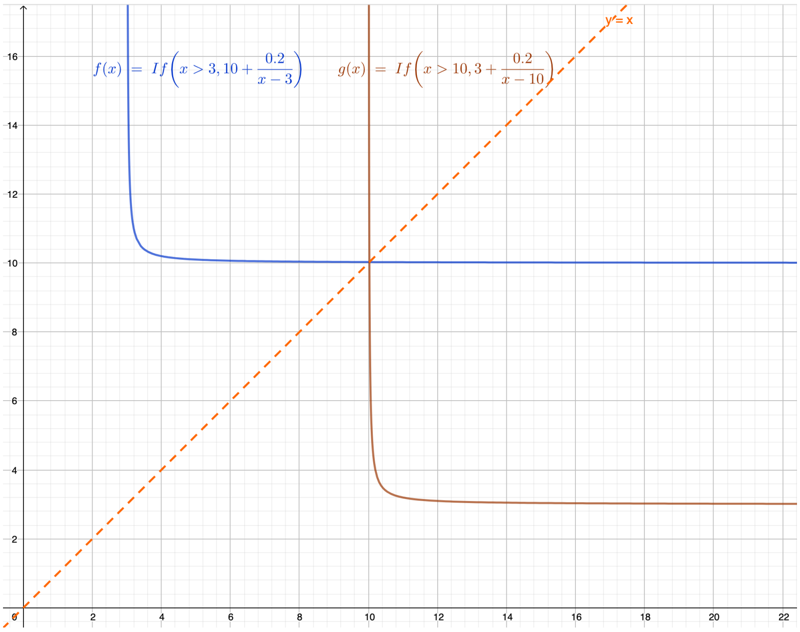

Figure 6: Inverse functions

b) We put $y = \ln(10x - x^2)$ and solve for x. Inserting both sides into $e^{(-)}$ we get $e^y = 10x - x^2$. Multiplication with -1 on both sides gives $x^2 - 10x = -e^y$. Completing the square on the left hand side gives $(x-5)^2 = 25 - e^y$. The two possibilities are $x = 5 \pm \sqrt{25 - e^y}$. Because $f(1) = \ln(9)$ we will have $1 = 5 \pm \sqrt{25 - e^{\ln 9}} = 5 \pm 4$. This is only true for minus. We can also get this more directly as $\sqrt{(x-5)^2} = |x-5| = -(x-5)$ because $x \leqslant 5$. Hence $\underline{\underline{g(x) = 5 - \sqrt{25 - e^x}}}$ (after switching variables). In general we have $D_g = V_f$. We see that $10x - x^2$ is strictly increasing for x in D_f and because $\ln(x)$ is a strictly increasing function, $f(x)$ is also strictly increasing. Hence the least function value is $f(1) = \ln(9)$ while the largest function value is $f(5) = \ln(25)$. This gives $D_g = V_f = \underline{\underline{[\ln(9), \ln(25)]}}$. Moreover $V_g = D_f = \underline{\underline{[1, 5]}}$.

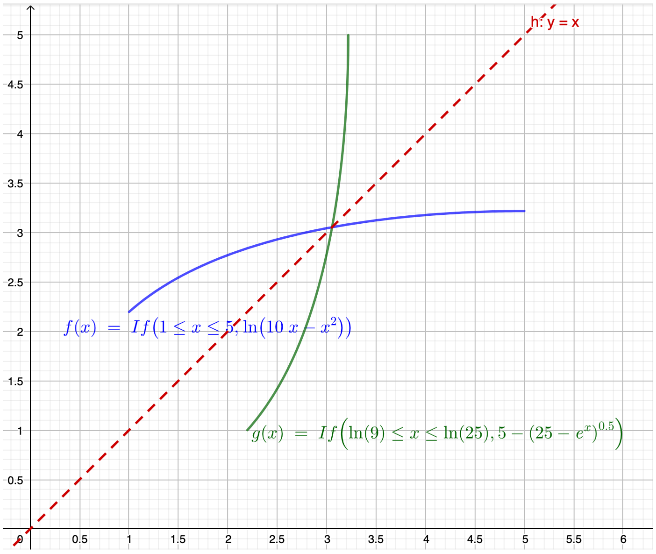

Figure 7: Inverse functions

Problem 10

We use the definition of a strictly increasing function: For all $x_1 < x_2$ one should have $f(x_1) < f(x_2)$. Hence assume $x_1 < x_2$. Then $x_2 - x_1 > 0$ and by the given fact we get $e^{x_2 - x_1} > 1$. Multiplying both sides with the positive number e^{x_1} we get $f(x_2) = e^{x_2} > e^{x_1} = f(x_1)$ which is what we had to show.