School exam (3h) EBA11802 - Mathematics for Data Science

8 Des. 2022

SOLUTIONS

Problem 1

i) We observe the centre of the ellipse as $(7, 2)$, the horizontal semi-axis as $14 - 7 = 7$ and the vertical semi-axis as $6 - 2 = 4$.

ii) Then the ellipse equation on standard form is

$$\frac{(x-7)^2}{49} + \frac{(y-2)^2}{16} = 1$$

Problem 2

Integer roots of $f(x)$ has to be divisors of 1. We find $f(-1) = -1 - 3 + 3 + 1 = 0$ so $x + 1$ is a factor in $f(x)$. Polynomial division gives

$$\begin{array}{l} (x^3 - 3x^2 - 3x + 1) : (x+1) = x^2 - 4x + 1 \\ \underline{-x^3 \; - x^2} \\ \quad -4x^2 - 3x \\ \quad \underline{\;\; 4x^2 + 4x} \\ \qquad\qquad x + 1 \\ \qquad\quad \underline{-x - 1} \\ \qquad\qquad\quad 0 \end{array}$$

The roots of $x^2 - 4x + 1$ are $2 \pm \sqrt{3}$ hence the factorisation is $f(x) = (x+1)(x - 2 + \sqrt{3})(x - 2 - \sqrt{3})$.

Problem 3

This is a «$\frac{0}{0}$»-expression and we therefore use l'Hôpital's rule and this repeats itself ones:

$$\lim_{x \to 0} \frac{0.5x + 1 - \sqrt{x+1}}{x^2} \overset{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{0.5 - \frac{1}{2\sqrt{x+1}}}{2x} \overset{\text{l'Hôp}}{=} \lim_{x \to 0} \frac{\frac{1}{4(x+1)\sqrt{x+1}}}{2} = \frac{1}{8}$$

Problem 4

i) Monthly interest is $3.6\%/12 = 0.3\%$ which gives monthly growth rate $1 + 0.3\%$. Annual growth rate is then $(1 + 0.3\%)^{12} = 1.0366$ so effective interest is 3.66%.

ii) If r is the interest the total present value of the cash flow is $-15 + \frac{30}{(1+r)^6}$. If r is the internal rate of return this present value should be 0 and that gives the equation $15 = \frac{30}{(1+r)^6}$, so $(1+r)^6 = 2$, that is $r = 2^{\frac{1}{6}} - 1 = 12.25\%$.

Problem 5

There are three conditions:

(1) $f(0) = 0 + 0 + 200 + 300 \cdot 1 = 500 > 0$

(2) $f'(x) = 0.06x + 5 + 3e^{0.01x} \geqslant 5 + 3 = 8 \geqslant 0$ for $x \geqslant 0$ so $f(x)$ is increasing for $x \geqslant 0$.

(3) $f''(x) = 0.06 + 0.03e^{0.01x} \geqslant 0.09 \geqslant 0$ for $x \geqslant 0$ so $f(x)$ is convex for $x \geqslant 0$.
Hence $f(x)$ is a cost function.

Problem 6

i) By the product and the chain rule we get
$D'(p) = e^{-0.05p} - 0.05(p+20)e^{-0.05p} = -0.05pe^{-0.05p}$. Then

$$\varepsilon(p) = \frac{-0.05p^2e^{-0.05p}}{(p+20)e^{-0.05p}} = \underline{\underline{\frac{-0.05p^2}{(p+20)}}}$$

ii) Because $\varepsilon(40) = \frac{-0.05 \cdot 40^2}{(40+20)} = \frac{-80}{60} = -\frac{4}{3} < -1$ the demand is price elastic at $p = 40$. Hence the income will decrease if the price increases a little from $p = 40$.

Problem 7

i) $f(x)$ is increasing in the interval $\langle \infty, 8]$, decreasing in the interval $[8, 16]$ and increasing in the interval $[16, \infty\rangle$. Moreover, $f(12) = 10$. It can give the following sketch (there are many):

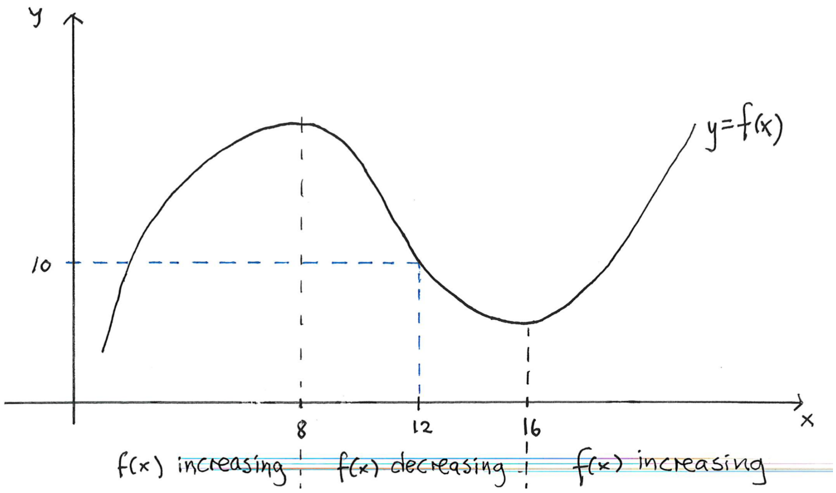

Figure 1: A possible graph of $f(x)$

ii) $g(x)$ is convex in the interval $\langle \infty, 30]$, concave in the interval $[30, 60]$ and convex in the interval $[60, \infty\rangle$. Moreover, the tangent of the graph is flat for $x = 60$. It can give the following sketch (there are many):

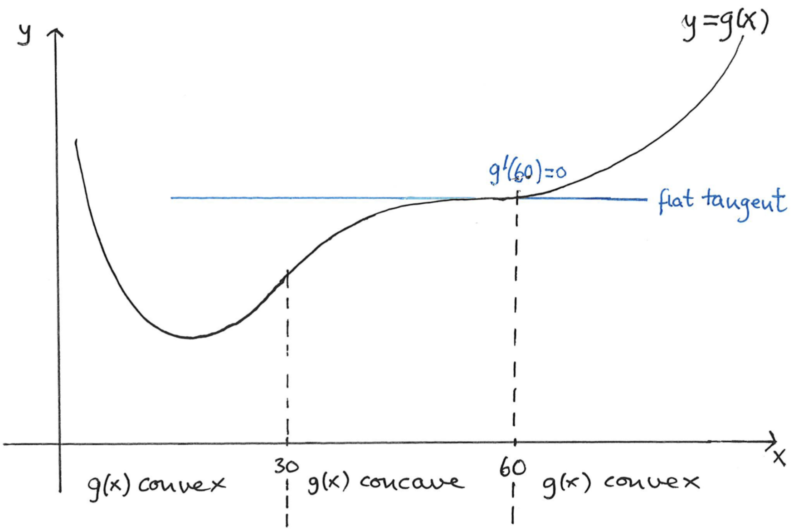

Figure 2: A possible graph of $g(x)$

Problem 8

i) False. $f(x)$ has stationary points where $f'(x)=0$, that is for $x\approx 1.3$, $x\approx 3.0$, $x\approx 8.2$ and $x\approx 12.7$ – hence four stationary points.

ii) False. $f(x)$ is strictly increasing in the interval $[4,8]$ because $f'(x)>0$ in this interval. Hence $f(4)<f(8)$.

iii) True. The inflection points of $f(x)$ is where $f''(x)$ changes sign, that is maximum and minimum points for $f'(x)$. We observe that the graph has three such points.

Problem 9

i) We find the stationary points of $f(x)$ which are in D_f. Applying the chain rule with $u(x)=x^2-20x+102$ and $g(u)=5\ln(u)$ we calculate $f'(x)=\frac{5(2x-20)}{x^2-20x+102}$. We see that the denominator is positive for all x because $x^2-20x+102=(x-10)^2+2\geqslant 2$. Hence the only stationary point is $x=10\in D_f$. Max. and min. is found either at the stationary point or in the endpoints of D_f (there are no cusps). We calculate $f(0)=5\ln(102)=23.12$, $f(10)=5\ln(2)=3.47$ and $f(25)=5\ln(15^2+2)=5\ln(227)=27.12$. Hence $x=10$ is the minimal point and $x=25$ is the maximal point of $f(x)$.

ii) We got the minimal value $f(10)=3.47$ and the maximal value $f(25)=27.12$.

Problem 10

The second degree Taylor polynomial of $f(x)$ at $x=10$ is $P_2(x)=200-3(x-10)+0.5(x-10)^2$. Hence $f(12)\approx P_2(12)=200-3(12-10)+0.5(12-10)^2=196$.

Problem 11

i) We always have $D_g=R_f$ og $R_g=D_f$.

$$\lim_{x\to\infty}\frac{2022e^x}{e^x+1}\overset{\text{l'Hôp}}{=}\lim_{x\to\infty}\frac{2022e^x}{e^x}=2022$$

and (without l'Hôpital)

$$\lim_{x\to-\infty}\frac{2022e^x}{e^x+1}=\frac{0}{1}=0$$

Since $f'(x)=\frac{2022e^x}{(e^x+1)^2}$ is positive for all x, $f(x)$ is strictly increasing. Hence $0<f(x)<2022$ and $D_g=\langle 0,2022\rangle$. Moreover, R_g the whole number line. An alternative is to write $f(x)=2022-\frac{2022}{(e^x+1)}$ (by «polynomial division»). On this form it is easier to see the horizontal asymptotes directly.

ii) We put $y=\frac{2022e^x}{e^x+1}$ and solve the equation for x. Multiply both sides by e^x+1 and get $ye^x+y=2022e^x$. Sorting terms with x on the left hand side and the rest on the right hand side first gives $(y-2022)e^x=-y$. Dividing with $(y-2022)$ on each side one obtains

$$e^x=\frac{-y}{y-2022}=\frac{y}{2022-y}$$

Because $D_g=\langle 0,2022\rangle$ both numerator and denominator are positive, so the fraction is positive. We can therefore put it into $\ln(-)$. It gives

$$x=\ln\left(\frac{y}{2022-y}\right)\qquad\text{which gives}\qquad g(x)=\ln\left(\frac{x}{2022-x}\right)=\ln(x)-\ln(2022-x)$$

Problem 12

i) Relative change is

$$\frac{\text{new price} - \text{old price}}{\text{old price}} = \underline{\underline{\frac{b-a}{a}}}$$

ii) If a is the price before the price changes then $a \cdot (1+r_1)$ is the price after the first price change. After the second price change the price is $a \cdot (1+r_1) \cdot (1+r_2)$ and after the last price change the price is $a \cdot (1+r_1) \cdot (1+r_2) \cdot (1+r_3) = b$. Hence

$$\underline{\underline{a = \frac{b}{(1+r_1) \cdot (1+r_2) \cdot (1+r_3)}}}$$