School exam (3h) EBA11802 - Mathematics for Data Science

5 May 2023

SOLUTIONS

Problem 1

We see that x is a common factor in all terms so $f(x) = x^4 - 7x^2 + 6x = x(x^3 - 7x + 6)$. Then we try and find that $x = 1$ is a zero for $x^3 - 7x + 6$, hence $x - 1$ is a factor in $x^3 - 7x + 6$. We calculate $(x^3 - 7x + 6) : (x - 1)$ by polynomial division:

$$
\begin{array}{l}
\big(\;x^3 \qquad -7x+6\big) : (x-1) = x^2 + x - 6 \\
\underline{-x^3 + x^2} \\
\qquad\quad x^2 - 7x \\
\qquad \underline{-x^2 \;\; + x} \\
\qquad\qquad -6x + 6 \\
\qquad\qquad\; \underline{6x - 6} \\
\qquad\qquad\qquad\;\; 0
\end{array}
$$

Hence $x^3 - 7x + 6 = (x - 1)(x^2 + x - 6)$ and because $x^2 + x - 6 = (x - 2)(x + 3)$ we get $\underline{\underline{f(x) = x(x - 1)(x - 2)(x + 3)}}$.

Problem 2

i) The expression for a hyperbola function can be written in the standard form $f(x) = c + \frac{a}{x-b}$ where the horizontal asymptote is $y = 100 = c$ and the vertical asymptote is $x = 30 = b$. Hence $f(40) = 100 + \frac{a}{40-30} = 99$ which gives $a = -10$ and $\underline{\underline{f(x) = 100 - \frac{10}{x-30}}}$.

ii) A function table should be provided.

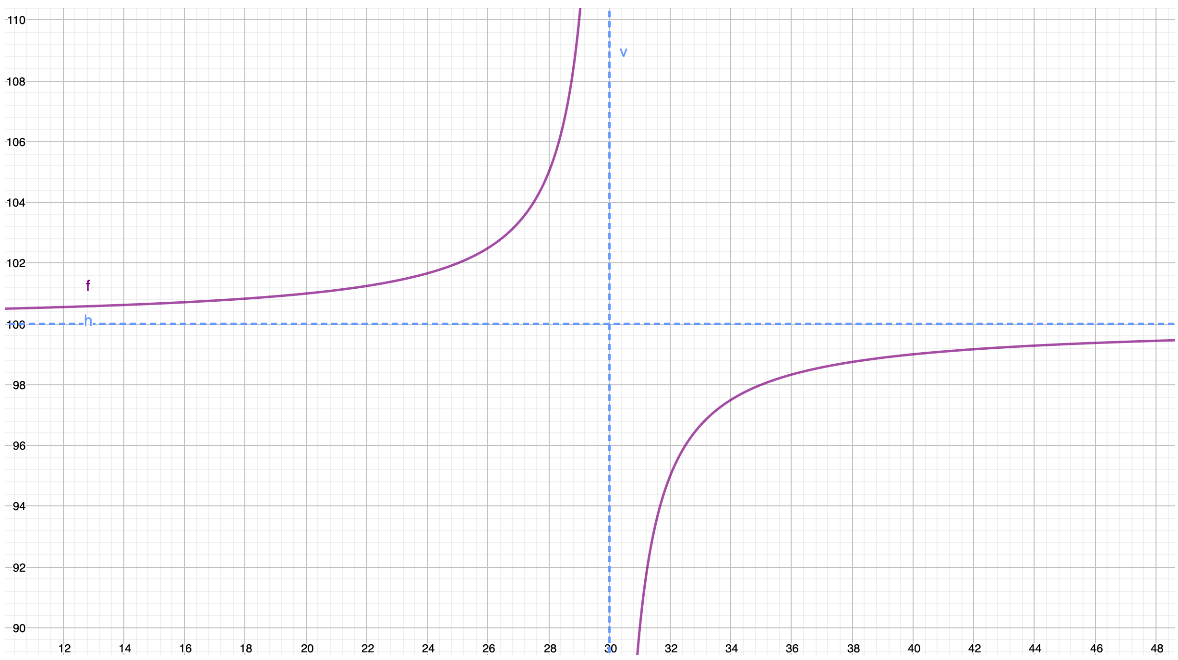

Figure 1: The graph of $f(x)$ with asymptotes

Problem 3

i) We write $f(x) = x^{1.5}$ and use the power rule: $f'(x) = \underline{\underline{1.5x^{0.5}}} = \underline{\underline{1.5\sqrt{x}}}$.

ii) We use the fraction rule: $f'(x) = \frac{3(x-1)-(3x-4)\cdot 1}{(x-1)^2} = \underline{\underline{\frac{1}{(x-1)^2}}}$

iii) We use the chain rule with $u = 2x+3$ and then the power rule and get:
$f'(x) = 2 \cdot 50 \cdot (2x+3)^{49} = \underline{\underline{100(2x+3)^{49}}}$

iv) A logarithm rule gives $\ln(xe^{-x}) = \ln(x) - x$ and hence $f(x) = x \cdot \ln(x) - x^2$. Then we use the product rule and the power rule and get $f'(x) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 2x = \underline{\underline{\ln(x) - 2x + 1}}$

Problem 4

a) The sum of the present values of each of the payments (in millions) is

$$\underline{\underline{\frac{2}{1.06^5} + \frac{2}{1.06^6} + \cdots + \frac{2}{1.06^{n+3}} + \frac{2}{1.06^{n+4}}}}$$

b) We read the geometric series backwards. The first term is $a_1 = \frac{2}{1.06^{n+4}}$, the multiplication factor is $k = 1.06$ and the number of terms is n. It gives the sum

$$\frac{2}{1.06^{n+4}} \cdot \frac{1.06^n - 1}{0.06}$$

$n = 20$ gives the present value (in millions)

$$\frac{2}{1.06^{24}} \cdot \frac{1.06^{20} - 1}{0.04} = \underline{\underline{18.17}}$$

c) If the cash flow continues forever we get the present value

$$a_1 \cdot \frac{1}{1-k} = \frac{2}{1.06^5} \cdot \frac{1}{1 - \frac{1}{1.06}} = \underline{\underline{26.40}}$$

Here we read the infinite geometric series from left to right with $k = \frac{1}{1.06}$ as multiplication factor.

Problem 5

i) We square each side and get $10 - x^2 = x^2 - 4x + 4$, that is $2x^2 - 4x = 6$, that is $x^2 - 2x = 3$. We complete the square and get $(x-1)^2 = 4$ which gives the candidate solutions $x = -1$, $x = 3$. But $x = -1$ gives the left hand side in the original equation equal to 3 while the right hand side gives -3. For $x = 3$ we get 1 on each side. Hence the only solution is $\underline{\underline{x = 3}}$.

ii) Here we use a rule for calculating logarithms and get the equation $\ln \frac{x+2}{x} \leqslant 0.1$. Insert both sides into $e^{(-)}$ (a strictly increasing function) and get the equivalent inequality $\frac{x+2}{x} \leqslant e^{0.1}$. Subtract $e^{0.1}$ from each side and make a common fraction. It gives

$$\frac{(1-e^{0.1})x + 2}{x} \leqslant 0$$

Because the original inequality is only defined for $x > 0$ we get the equivalent inequality $(1-e^{0.1})x + 2 \leqslant 0$, that is $(1-e^{0.1})x \leqslant -2$. We divide with the negative number $(1-e^{0.1})$ on each side and get $\underline{\underline{x \geqslant \frac{2}{e^{0.1}-1} \approx 19.02}}$.

Problem 6

i) By the chain rule we get that the marginal cost function is
$K'(x) = \underline{\underline{0.05 \cdot K_0 \cdot e^{0.05x}}} = \underline{\underline{0.05 \cdot K(x)}}$.

ii) Since the cost function is strictly convex, cost optimum is the solution of the equation $A(x) = K'(x)$ where $A(x) = \frac{K(x)}{x}$ is the average unit cost. That is $\frac{K(x)}{x} = 0.05 \cdot K(x)$. We divide with $K(x)$ on each side and get $\frac{1}{x} = 0.05$, that is $x = \frac{1}{0.05} = \underline{\underline{20}}$.
Optimal unit cost is the unit cost when one produces cost optimum many units. From the equation we get $A(20) = K'(20) = 0.05 \cdot K_0 \cdot e^{0.05 \cdot 20} = \underline{\underline{0.05e \cdot K_0}} \approx \underline{\underline{0.1359 \cdot K_0}}$.

Problem 7

i) We have $f'(x) = (16x^2 - 25)/x$. We have $16x^2 - 25 \leqslant 0$ if and only if $-\frac{5}{4} \leqslant x \leqslant \frac{5}{4}$. But $f(x)$ is only defined for $x > 0$ so $f'(x) \leqslant 0$ for $0 < x \leqslant \frac{5}{4}$ and $f'(x) \geqslant 0$ for $x \geqslant \frac{5}{4}$. Hence $f(x)$ is decreasing in the interval $\langle 0, \frac{5}{4}]$ and increasing in the interval $[\frac{5}{4}, \rightarrow\rangle$.

ii) We have $f''(x) = (16x^2 + 25)/x^2 > 0$ for all $x > 0$. Hence $f(x)$ is convex for all $x > 0$.

Problem 8

i) $f(x)$ has stationary points where $f'(x) = 0$, that is where the tangent is horizontal. In the interval $[3, 10]$ that is at (approximately) $x = 4.8$ and $x = 7.5$. The statement is false.

ii) $f(x)$ has inflection points where $f''(x)$ changes sign, that is at the border between convex and concave. In the interval $\langle 4, 24]$ this happens (approximately) at $x = 6$, $x = 9.8$ and at $x = 17.8$. The statement is true.

iii) $f'(x)$ is decreasing where $f''(x) \leqslant 0$, that is where $f(x)$ is concave. In the interval $[12, 24]$ the function $f(x)$ is first concave until 17.6, then convex. Hence $f'(x)$ is not decreasing in the whole interval . The statement is false.

Problem 9

i) By the power rule $D'(p) = 30 \cdot (-0.8)p^{-1.8}$. Then

$$\varepsilon(p) = \frac{D'(p) \cdot p}{D(p)} = \frac{30 \cdot (-0.8)p^{-1.8} \cdot p}{30p^{-0.8}} = \underline{\underline{-0.8}} \quad \text{(a constant function)}$$

ii) Because $\varepsilon(20) = -0.8 > -1$ the demand function is inelastic at $p = 20$. Hence the revenue increases if the price increases somewhat from $p = 20$. This can also be seen directly since the revenue function $R(p) = p \cdot D(p) = 30p^{0.2}$ is an increasing function.

Problem 10

i) We put $y = e^{-0.02x} + 100$ and solve for x. That gives $e^{-0.02x} = y - 100$. By inserting both sides into $\ln(-)$ we get $-0.02x = \ln(y - 100)$. Multiplying with -50 on each side gives $y = -50\ln(y - 100)$. Changing variables gives $\underline{\underline{g(x) = -50\ln(x - 100)}}$.

ii) We always have $D_g = R_f$. Because $f(x)$ is a strictly decreasing function the greatest function value is er $f(0) = 101$. We also see that

$$f(x) = 100 + 1/e^{0.02x} \underset{x \to \infty}{\longrightarrow} 100^+$$

Hence $D_g = R_f = \underline{\underline{\langle 100, 101]}}$. We always have $V_g = D_f$ so $\underline{\underline{V_g = [0, \infty\rangle}}$.

Problem 11

i) With continuous compounding the annual growth factor is $e^{0.072}$. Hence the effective interest is $r_{\text{eff}} = e^{0.072} - 1 = 7.47\%$.

ii) If r is the interest then the present value of the cash flow is $-A + B/(1+r)^5$. Since r is supposed to be the internal rate of return the present value has to be 0, that is $B/(1+r)^5 = A$, that is $(1+r)^5 = B/A$, that is $1 + r = (B/A)^{1/5}$. Hence the internal rate of return is $r = (B/A)^{1/5} - 1$.

We could also assume continuous compounding. Then the present value is $-A + B \cdot e^{-5r}$. It gives the equation $B \cdot e^{-5r} = A$ for the internal rate of return, that is $e^{5r} = B/A$. Hence the internal rate of return is $r = \frac{1}{5}\ln(B/A)$.

Problem 12

i) We think of y as an (undetermined) function of x and differentiate each side of the equation with respect to x. It gives $3x^2 - 4y - 4xy' + 2yy' = 0$. Here we have used both the product rule and the chain rule. Then we solve the equation for y'. Collect terms with y' on the left hand side and factorise: $(2y - 4x)y' = 4y - 3x^2$. Dividing with $(2y - 4x)$ on each side gives

$$y' = \frac{4y - 3x^2}{2y - 4x}$$

ii) We use the one point formula $y - 3 = a(x - 1)$ where a is the slope given by the expression for y', that is $a = (4 \cdot 3 - 3 \cdot 1^2)/(2 \cdot 3 - 4 \cdot 1) = 9/2$. Hence the tangent equation $y - 3 = 4.5(x - 1)$ gives $y = 4.5x - 1.5$.