Suggested solution	EBA 1180 Mathematics for Data Science
Date	May 24th 2023 from 09.00 - 14.00

Question 1.

(a) We write down the extended matrix of the system, and use elementary row operations:

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 2 & 1 & 1 & 0 & 2 \\ 4 & 2 & 1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 0 & 3 & -5 & -8 & -20 \\ 0 & 6 & -11 & -14 & -44 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 3 & 4 & 11 \\ 0 & 3 & -5 & -8 & -20 \\ 0 & 0 & -1 & 2 & -4 \end{array}\right)$$

The result is an echelon form, and we have marked the pivot positions in blue. Hence, there are infinitely many solutions, with w free when we write $\mathbf{x} = (x,y,z,w)$ for the unknowns. We find the solutions by back substitution: From the final equation, we get $-z + 2w = -4$, or $z = 2w + 4$. The next equations gives $3y - 5(2w+4) - 8w = -20$, or $3y = 18w$, that is $y = 6w$. The first equations gives $x - (6w) + 3(2w+4) + 4w = 11$, or $x = -4w - 1$. Hence, the solutions of the linear system can be written

$$\mathbf{x} = \begin{pmatrix} -4w-1 \\ 6w \\ 2w+4 \\ w \end{pmatrix} = w \begin{pmatrix} -4 \\ 6 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 4 \\ 0 \end{pmatrix}$$

where w is a free variable.

(b) If we replace $\mathbf{b}$ with $\mathbf{0}$ in (a), we find that the solutions of the linear system $A\mathbf{x} = \mathbf{0}$ are given by $\mathbf{x} = w \cdot (-4,6,2,1)$ where w is a free variable. In particular, $w = 1$ gives the solution $\mathbf{x} = (-4,6,2,1)$ of the linear system, which means that $-4\mathbf{v}_1 + 6\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$. We can solve this vector equation for $\mathbf{v}_3$ and find that

$$-2\mathbf{v}_3 = -4\mathbf{v}_1 + 6\mathbf{v}_2 + \mathbf{v}_4 \quad \Rightarrow \quad \mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_4$$

Alternatively, we could solve the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_4\mathbf{v}_4 = \mathbf{v}_3$ by writing it as a linear system and using Gaussian elimination.

Question 2.

(a) Since $(e^{2x})' = 2e^{2x}$ by the chain rule, we get $\int e^{2x}\, \mathrm{d}x = e^{2x}/2 + C$, and hence

$$\int_0^1 1 + e^{2x}\, \mathrm{d}x = \left[x + \frac{1}{2}e^{2x} \right]_0^1 = 1 + \frac{1}{2}e^2 - \frac{1}{2}e^0 = \frac{1}{2}\left(e^2 + 1\right)$$

(b) We use the substitution $u = x + 1$, with $\mathrm{d}u = \mathrm{d}x$, and the power rule for integration:

$$\int 15x\sqrt{x+1}\, \mathrm{d}x = \int 15(u-1)u^{1/2}\, \mathrm{d}u = \int 15u^{3/2} - 15u^{1/2}\, \mathrm{d}u = 6u^{5/2} - 10u^{3/2} + C$$

This gives

$$\int_0^1 15x\sqrt{x+1}\, \mathrm{d}x = \left[6u^{5/2} - 10u^{3/2} \right]_1^2 = 6(4\sqrt{2} - 1) - 10(2\sqrt{2} - 1) = 4\sqrt{2} + 4$$

(c) We use partial fractions to simplify the integrand:

$$\frac{3}{9 - x^2} = \frac{A}{3 - x} + \frac{B}{3 + x} \quad \Rightarrow \quad 3 = A(3 + x) + B(3 - x)$$

This gives $(A - B)x + (3A + 3B) = 3$, and hence $A - B = 0$ and $3A + 3B = 3$. From this, we get that $A = B$ and $6A = 3$, or $A = 1/2$. The integral becomes

$$\int \frac{3}{9 - x^2}\, \mathrm{d}x = \int \frac{1/2}{3 - x} + \frac{1/2}{3 + x}\, \mathrm{d}x = -\frac{1}{2}\ln|3 - x| + \frac{1}{2}\ln|3 + x| + C = \frac{1}{2}\ln\left|\frac{3 + x}{3 - x}\right| + C$$

This gives

$$\int_0^1 \frac{3}{9 - x^2}\, \mathrm{d}x = \left[\frac{1}{2}\ln\left|\frac{3 + x}{3 - x}\right| \right]_0^1 = \frac{1}{2}\ln(4/2) = \frac{1}{2}\ln 2$$

(d) Note that $2x\ln(\sqrt{x}) = 2x \cdot \frac{1}{2}\ln x = x\ln x$ since $\sqrt{x} = x^{1/2}$. Then, we use integration by parts with $u' = x$ and $v = \ln(x)$, which gives $u = x^2/2$ and $v' = 1/x$. Since $\int u'v \, dx = uv - \int uv' \, dx$, integration by parts gives

$$\int x\ln(x) \, dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \, dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x \, dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

This gives

$$\int 2x\ln(\sqrt{x}) \, dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

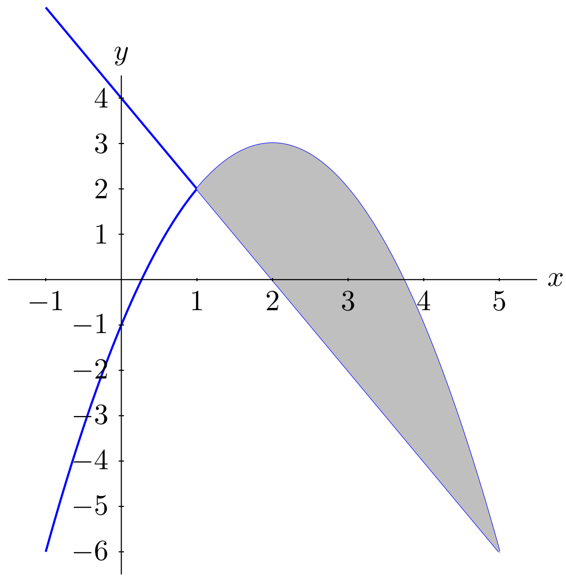
(e) The parabola P is the graph of the function $f(x) = a\left[(x-2)^2 - 3\right]$ since $x = 2 \pm \sqrt{3}$ are the zeros of f, and since it intersects the y-axis in $y = -1$, we have $f(0) = a(4-3) = -1$, so $a = -1$.Hence, $f(x) = 3 - (x-2)^2$. The straight line L is the graph of the function $g(x) = -2x + b$ since it has slope -2, and $b = 4$ since $f(1) = 2$ and $g(1) = -2 + b$. Hence, we get $g(x) = 4 - 2x$. The intersection points are given by

$$3 - (2-x)^2 = 4 - 2x \quad \Rightarrow \quad -x^2 + 6x - 5 = 0$$

The intersection points become $x = 1$ and $x = 5$. The part of the plane (in grey) is shown in the figure below. The area of the grey part of the plane is

$$A = \int_1^5 f(x) - g(x) \, dx = \int_1^5 -x^2 + 6x - 5 \, dx = \left[-\frac{1}{3}x^3 + 3x^2 - 5x\right]_1^5$$

$$= \left(-\frac{125}{3} + 75 - 25\right) - \left(-\frac{1}{3} + 3 - 5\right) = -\frac{124}{3} + 50 + 2 = \frac{156 - 124}{3} = \frac{32}{3}$$



Question 3.

(a) We use cofactor expansion along the first row to compute the determinant:

$$\begin{vmatrix} t & 2 & 4 \\ 2 & t & 4 \\ 2 & 4 & t \end{vmatrix} = t(t^2 - 16) - 2(2t - 8) + 4(8 - 2t) = (t-4)(t(t+4) - 4 - 8)$$

$$= (t-4)(t^2 + 4t - 12) = (t-4)(t+6)(t-2) = (t-2)(t-4)(t+6)$$

Here, we have used that $t^2 - 16 = (t-4)(t+4)$ and that $t - 4$ is a common factor in the cofactor expansion. We can also write the determinant as $|A| = t^3 - 28t + 48$.

(b) When $t = 1$ we get $\det(A) = (-1)(-3)7 = 21 \neq 0$, hence A has an inverse matrix given by

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & C_13 \\ C_{21} & C_{22} & C_23 \\ C_{31} & C_{32} & C_33 \end{pmatrix}^T$$

where C_{ij} is the cofactor of A in position (i,j). With $t = 1$, the inverse matrix is given by

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{21} \begin{pmatrix} -15 & 6 & 6 \\ 14 & -7 & 0 \\ 4 & 4 & -3 \end{pmatrix}^T = \frac{1}{21} \begin{pmatrix} -15 & 14 & 4 \\ 6 & -7 & 4 \\ 6 & 0 & -3 \end{pmatrix}$$

(c) The linear system has one unique solution when $|A| \neq 0$, and none or infinitely many solutions when $|A| = 0$. Since $|A| = 0$ for $t = 2, 4, -6$, we see that the linear system has precicely one solution for all values of t except for $t = 2$, $t = 4$ and $t = -6$.

Question 4.

(a) The partial derivatives of $f(x,y) = x^2y + xy^2 - 3xy$ are $f'_x = 2xy + y^2 - 3y$ and $f'_y = x^2 + 2xy - 3x$, and the first order conditions $f'_x = f'_y = 0$ are given by

$$\begin{aligned} y(2x + y - 3) = 0 \quad &\Rightarrow \quad y = 0 \text{ eller } 2x + y = 3 \\ x(x + 2y - 3) = 0 \quad &\Rightarrow \quad x = 0 \text{ eller } x + 2y = 3 \end{aligned}$$

This gives four cases: $x = y = 0$, $y = 0,\ x + 2y = 3$, $x = 0,\ 2x + y = 3$, or $2x + y = x + 2y = 3$. The first three cases gives the points $(x,y) = (0,0), (0,3), (3,0)$, and the final case gives $(x,y) = (1,1)$. In the final case, we can for example use Gaussian elimination to find the solution. We conclude that we have four stationary points for f:

$$(x,y) = (0,0), (0,3), (3,0), (1,1)$$

(b) The Hessian matrix of f in an arbitrary point is given by

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 2y & 2x + 2y - 3 \\ 2x + 2y - 3 & 2x \end{pmatrix}$$

We insert the first three stationary points $(x,y) = (0,0), (0,3), (3,0)$ into the Hessian and get

$$H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}, \quad H(f)(3,0) = \begin{pmatrix} 0 & 3 \\ 3 & 6 \end{pmatrix}, \quad H(f)(0,3) = \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}$$

In all three cases, $\det H(f) = -9 < 0$, and hence the points $(0,0), (3,0), (0,3)$ are saddle points for f. In the point $(1,1)$ we get the Hessian

$$H(f)(1,1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Since $\det H(f)(1,1) = 4 - 1 = 3 > 0$ and $\operatorname{tr} H(f)(1,1) = 2 + 2 = 4 > 0$, the point $(1,1)$ is a local minimum for f. Since f does not have local maximum points, the function f does not have a maximum value. The only candidate for a minimum value for f is found in the local minimum point $(1,1)$ with function value $f(1,1) = -1$. But since $f(-2,-2) = -8 - 8 - 12 = -28 < -1$, this is not a global minimum for f. We conclude that f does not have a maximum nor a minimum value.

Question 5.

(a) We use the Lagrange multiplier method with the Lagrangian $\mathcal{L} = xy - \lambda(x^2 + y^2 + x^2y^2 - 3)$ to find candidate points. The Lagrange conditions are

$$\begin{aligned} \mathcal{L}'_x = y - \lambda(2x + 2xy^2) &= 0 \\ \mathcal{L}'_y = x - \lambda(2y + 2x^2y) &= 0 \\ x^2 + y^2 + x^2y^2 &= 3 \end{aligned}$$

We solve the first two equations for λ. This gives

$$\lambda = \frac{y}{2x(1 + y^2)} = \frac{x}{2y(1 + x^2)}$$

We have used the factorizations $2x + 2xy^2 = 2x(1 + y^2)$ and $2y + 2x^2y = 2y(1 + x^2)$ in the previous expressions. Note that one of the denominators become zero if $x = 0$ or $y = 0$. But $x = 0$ gives $y = 0$ in the first condition, and $y = 0$ gives $x = 0$ from the second condition. Since

$(x,y) = (0,0)$ does not satisfy the constraint, we do not lose any solutions by using the fractional expressions above. By cross multiplying (or multiplication by a common denominator) we get

$$y \cdot 2y(1 + x^2) = x \cdot 2x(1 + y^2) \quad \Rightarrow \quad 2y^2 + 2y^2x^2 = 2x^2 + 2x^2y^2 \quad \Rightarrow \quad 2y^2 = 2x^2$$

Hence, $y^2 = x^2$. If we insert this into the constraint, we get $x^2+x^2+x^4 = 3$, or $x^4+2x^2-3 = 0$. This can be written

$$x^4 + 2x^2 - 3 = (x^2 + 3)(x^2 - 1) = 0 \quad \Rightarrow \quad x^2 = 1$$

since $x^2 + 3 > 0$ for all x. Hence, $x = \pm 1$, and $y^2 = x^2$ so $y = \pm 1$, and for each of the four combinations of signs, we find λ from the fractional expression above. Hence, we get that the following four points satisfy the Lagrange conditions:

$$(x,y;\lambda) = (1,1;1/4), (-1,-1;1/4), (1,-1;-1/4), (-1,1;-1/4)$$

(b) A point has a degenerate constraint if $g'_x = g'_y = 0$, where $g(x,y) = x^2 + y^2 + x^2y^2$. This gives

$$g'_x = 2x + 2xy^2 = 0,\ g'_y = 2y + 2x^2y = 0 \quad \Rightarrow \quad 2x(1 + y^2) = 2y(1 + x^2) = 0$$

Since $1 + y^2, 1 + x^2 > 0$ for all x,y, this means that $x = y = 0$. But the point $(x,y) = (0,0)$ is not admissible since it does not satisfy the constraint; $g(0,0) = 0 \neq 3$. We conclude that there are no admissible points with degenerate constraint for this problem.

(c) Note that the set D of admissible points, given by the equation $g(x,y) = x^2 + y^2 + x^2y^2 = 3$, is a compact set: It is closed since it is given by an equality, and it is bounded because $-\sqrt{3} \leq x,y \leq \sqrt{3}$ for all points (x,y) in D. We can see this in the following way: Since $x^2 + y^2 + x^2y^2 = 3$ and each of the terms $x^2, y^2, x^2y^2 \geq 0$ since they are squares, hence $x^2, y^2, x^2y^2 \leq 3$. From $x^2 \leq 3$ it follows that $-\sqrt{3} \leq x \leq \sqrt{3}$, and similarly, it follows from $y^2 \leq 3$ that $-\sqrt{3} \leq x \leq \sqrt{3}$. Since f is continuous, it follows from the Extreme Value Theorem that the Lagrange problem has a maximum. This maximum point must be one of the candidate points we found in (a) since there are no admissible points with degenerate constraint. Since $f(1,1) = f(-1,-1) = 1$ and $f(-1,1) = f(-1,1) = -1$, it follows that the maximum value is $f_{\max} = 1$ in the points $(x,y) = (1,1), (-1,-1)$ with $\lambda = 1/4$.