Exam	EBA 1180 Mathematics for Data Science
Date	May 24th 2023 from 09.00 to 14.00

The exam consists of 15 problems with equal weight. All answers must be justified.

Question 1.
Consider the matrix A and the vector $\mathbf{b}$ given by

$$A = \begin{pmatrix} 1 & -1 & 3 & 4 \\ 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 11 \\ 2 \\ 0 \end{pmatrix}.$$

Let the column vectors of A be called $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ respectively.

(a) Use Gaussian elimination to solve the linear system $A\mathbf{x} = \mathbf{b}$.
(b) Write $\mathbf{v}_3$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4$ if this is possible.

Question 2.
Compute the following integrals:

a) $\displaystyle\int_0^1 1 + e^{2x}\,\mathrm{d}x$ b) $\displaystyle\int_0^1 15x\sqrt{x+1}\,\mathrm{d}x$ c) $\displaystyle\int_0^1 \frac{3}{9-x^2}\,\mathrm{d}x$ d) $\displaystyle\int 2x\ln(\sqrt{x})\,\mathrm{d}x$

The parabola P intersects the x-axis in $x = 2 \pm \sqrt{3}$ and the y-axis in $y = -1$. The straight line L has slope -2 and intersects P in $x = 1$.

e) Find the area of the part of the plane which is bounded by P and L. Make a figure to illustrate this.

Question 3.
Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{pmatrix} t & 2 & 4 \\ 2 & t & 4 \\ 2 & 4 & t \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) Compute $|A|$.
(b) Find A^{-1} when $t = 1$.
(c) Determine all values of t for which $A\mathbf{x} = \mathbf{b}$ has precisely one solution.

Question 4.
Consider the function f given by $f(x,y) = x^2y + xy^2 - 3xy$.

(a) Find the stationary points of f.
(b) Classify these stationary points. Does f have a maximum- or minimum value (or both)?

Question 5.
Consider the Lagrange problem $\max f(x,y) = xy$ when $x^2 + y^2 + x^2y^2 = 3$.

(a) Write down the three Lagrange conditions, and find all points $(x,y;\lambda)$ that satisfy these.
(b) Are there any admissible points with degenerate constraint for this problem?
(c) Determine whether the Lagrange problem has a maximum value, and if so, find this maximum value.

Formula Sheet

Financial mathematics

Geometric series.
A finite geometric series has sum

$$S_n = a_1 \cdot \frac{1-k^n}{1-k} = a_1 \cdot \frac{k^n - 1}{k-1}$$

and an infinite geometric series has sum

$$S = a_1 \cdot \frac{1}{1-k} \quad \text{when } |k| < 1$$

Present values.
The present value K_0 of a payment K is given by

$$K_0 = \frac{K_n}{(1+r)^n} \quad \text{and} \quad K_0 = \frac{K_n}{e^{rn}}$$

using discrete and continuous compounding.

Integration

Integration techniques.

a) Integration by parts:

$$\int u'v \, \mathrm{d}x = uv - \int uv' \, \mathrm{d}x$$

b) Substitution:

$$\int f(u)u' \, \mathrm{d}x = \int f(u) \, \mathrm{d}u$$

c) Partial fractions:

$$\int \frac{px+q}{(x-a)(x-b)} \, \mathrm{d}x = \int \left(\frac{A}{x-a} + \frac{B}{x-b} \right) \mathrm{d}x$$

Area.
The area of the region bounded by $a \le x \le b$ and $f(x) \le y \le g(x)$ is given by

$$A = \int_a^b (g(x) - f(x)) \, \mathrm{d}x$$

Linear algebra

Cramer's rule.
A linear system $A\mathbf{x} = \mathbf{b}$ where $|A| \neq 0$ has a unique solution given by

$$x_1 = \frac{|A_1(\mathbf{b})|}{|A|} \quad x_2 = \frac{|A_2(\mathbf{b})|}{|A|} \quad \dots \quad x_n = \frac{|A_n(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained by replacing column i of A by $\mathbf{b}$.

Functions of two variables

Second derivative test.
A stationary point (x^*, y^*) of the function $f(x,y)$ is a

a) local minimum if $A > 0$ and $AC - B^2 > 0$
b) local maximum if $A < 0$ and $AC - B^2 > 0$
c) saddle point if $AC - B^2 < 0$

when $H(f)(x^*, y^*) = \left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)$.

Level curves.
The slope $y' = \mathrm{d}y/\mathrm{d}x$ of the tangent line to the level curve $f(x,y) = c$ is given by

$$y' = -\frac{f'_x}{f'_y}$$

Method of Lagrange multipliers.
The Lagrange conditions for the problem

$$\max / \min f(x,y) \text{ when } g(x,y) = a$$

is given by

$$\mathcal{L}'_x = 0, \ \mathcal{L}'_y = 0, \ g(x,y) = a$$

An admissible point has degenerated constraint if

$$g'_x = 0, \ g'_y = 0$$