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| Solutions | EBA 1180 Mathematics for Data Science |
| Date | December 18th 2023 at 0900 - 1400 |

**Question 1.**

(a) We write down the extended matrix of the system, and use elementary row operations:

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 5 & 7 & 14 \\ 1 & 2 & 4 & 4 & 10 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 3 & 1 & 6 \\ 0 & 0 & 3 & 1 & 6 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 3 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The result is an echelon form, and we have marked the pivot positions in blue. Hence, there are infinitely many solutions, with $y$ and $w$ free when we write $\mathbf{x} = (x,y,z,w)$ for the unknown. We find the solution by back substitution, where we ignore the row of zeros at the bottom, as this gives a trivial equation: From the second equation, we find $3z+w = 6$, or $z = (6-w)/3 = 2-w/3$. The first equation gives $x+2y+z+3w = 4$, or $x = 4-2y-(2-w/3)-3w = 2-2y-8w/3$. Hence, the solution of the linear system can be written

$$\mathbf{x} = \begin{pmatrix} 2-2y-8w/3 \\ y \\ 2-w/3 \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{w}{3} \begin{pmatrix} -8 \\ 0 \\ -1 \\ 3 \end{pmatrix}$$

where $y,w$ are free variables.

(b) We know that $\mathbf{w}$ is a linear combination of the column vectors of $A$ if and only if the linear system $A\mathbf{x} = \mathbf{w}$ has solutions. We repeat the row operations above with $\mathbf{b}$ replaced by $\mathbf{w}$:

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & a \\ 2 & 4 & 5 & 7 & b \\ 1 & 2 & 4 & 4 & c \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & a \\ 0 & 0 & 3 & 1 & b-2a \\ 0 & 0 & 3 & 1 & c-a \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 3 & a \\ 0 & 0 & 3 & 1 & b-2a \\ 0 & 0 & 0 & 0 & (c-a)-(b-2a) \end{array}\right)$$

Since $(c-a)-(b-2a) = a-b+c$, we find that the linear system has infinitely many solutions (two degrees of freedom) if $a-b+c = 0$, and no solutions otherwise (since in this case, we have a pivot in the final column). Hence, $\mathbf{w}$ is a linear combination of the column vectors in $A$ for the values of $(a,b,c)$ where $a-b+c = 0$.

**Question 2.**

(a) We use integration by parts with $u' = 4x$ and $v = \ln x$. Hence, we get that $u = 2x^2$ and $v' = 1/x$. Based on this, we can calculate the indefinite integral

$$\int 4x \ln x \, \mathrm{d}x = 2x^2 \ln x - \int 2x^2 \cdot \frac{1}{x} \, \mathrm{d}x = 2x^2 \ln x - \int 2x \, \mathrm{d}x = 2x^2 \ln x - x^2 + C$$

Hence, the definite integral is

$$\int_1^2 4x \ln x \, \mathrm{d}x = \left[2x^2 \ln x - x^2\right]_1^2 = 8\ln 2 - 4 - (-1) = 8\ln 2 - 3$$

(b) We use the substitution $u = x+1$, with $\mathrm{d}u = \mathrm{d}x$ and $x = u-1$, and the power rule for integration to get that

$$\int_0^1 \frac{3x}{\sqrt{x+1}} \, \mathrm{d}x = \int_1^2 \frac{3(u-1)}{\sqrt{u}} \, \mathrm{d}u = 3\int_1^2 u^{1/2} - u^{-1/2} \, \mathrm{d}u = 3\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^2$$

since the new bounds of the definite integral are given from that $x = 0$ implies $u = 1$ and $x = 1$ implies $u = 2$. Hence,

$$\int_0^1 \frac{3x}{\sqrt{x+1}} \, \mathrm{d}x = \left[2u^{3/2} - 6u^{1/2}\right]_1^2 = 4\sqrt{2} - 6\sqrt{2} - 2 - (-6) = 4 - 2\sqrt{2}$$

(c) The factorization $x^2 - 5x + 6 = (x-2)(x-3)$ of the denominator can be used for partial fractions:

$$\frac{x}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3} \quad \Rightarrow \quad x = A(x-3) + B(x-2)$$

This implies $(A+B)x + (-3A-2B) = x$, and hence $A+B=1$ and $-3A-2B=0$. Hence, we get $B=3$ (for instance by adding 3 times the first equation to the second equation), and hence, $A=-2$. This gives the following integral

$$\int_0^1 \frac{x}{x^2-5x+6}\,\mathrm{d}x = \int_0^1 \frac{-2}{x-2} + \frac{3}{x-3}\,\mathrm{d}x = [-2\ln|x-2| + 3\ln|x-3|]_0^1$$
$$= (-2\ln 1 + 3\ln 2) - (-2\ln 2 + 3\ln 3) = 5\ln 2 - 3\ln 3$$

(d) We solve $\int e^{\sqrt{x}}\,\mathrm{d}x$ by the substitution $u=\sqrt{x}$, which gives $\mathrm{d}u = u'\,\mathrm{d}x$ with $u' = 1/(2\sqrt{x})$. This implies

$$\int e^{\sqrt{x}}\,\mathrm{d}x = \int e^u \cdot (2\sqrt{x})\,\mathrm{d}u = \int e^u \cdot 2u\,\mathrm{d}u = \int 2u\,e^u\,\mathrm{d}u$$

To solve this integral, we use integration by parts with $v' = e^u$ and $w = 2u$, which gives $v = e^u$ and $w' = 2$ (we use the symbols $v$ and $w$ instead of $u$ and $v$, since $u$ has already been used in the substitution):

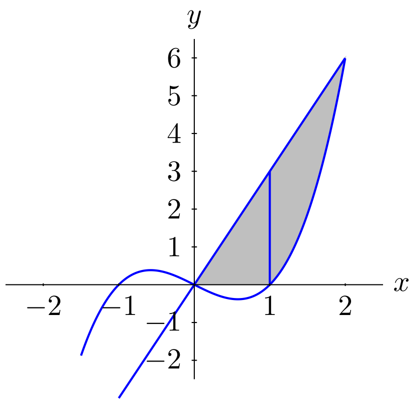
$$\int 2ue^u\,\mathrm{d}u = 2ue^u - \int 2\cdot e^u\,\mathrm{d}u = 2ue^u - 2e^u + \mathcal{C} = (2\sqrt{x}-2)e^{\sqrt{x}} + \mathcal{C}$$

(e) The graph of $f(x) = x^3 - x$ has zeros given by $x^3 - x = x(x^2-1) = 0$ which gives $x = -1, 0, 1$. The graph is underneath the $x$-axis in the interval (0,1) and above the $x$-axis for $x > 1$. The straight line L has equation $y = 3x$ and the intersection with the graph of $f$ is given by

$$x^3 - x = 3x \quad \Rightarrow \quad x^3 - 4x = x(x^2-4) = 0$$

Hence, the intersections are $x=-2$, $x=0$, and $x=2$. Therefore, the part of the plane R is between the line L and the $x$-axis in the interval [0,1], and between the line L and the graph of $f$ in the interval [1,2]. The part of the plane is shown (in color) in the figure below, and the area of R is given by

$$A(R) = \int_0^1 3x\,\mathrm{d}x + \int_1^2 3x - (x^3 - x)\,\mathrm{d}x = \left[\frac{3}{2}x^2\right]_0^1 + \int_1^2 4x - x^3\,\mathrm{d}x$$
$$= \frac{3}{2} + \left[2x^2 - \frac{1}{4}x^4\right]_1^2 = \frac{3}{2} + (8-4) - (2 - \frac{1}{4}) = 3 + \frac{1}{2} + \frac{1}{4} = \frac{15}{4} = 3.75$$




**Question 3.**

(a) We use cofactor expansion along the first row to compute the determinant:

$$\begin{vmatrix} t & 1 & t \\ 1 & t & 2 \\ t & 2 & t \end{vmatrix} = t(t^2-4) - 1(t-2t) + t(2-t^2) = t^3 - 4t + t + 2t - t^3 = -t$$

(b) When $t=1$, we get $\det(A) = -1 \neq 0$. Hence, $A$ has an inverse matrix given by

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

where $C_{ij}$ is the cofactor of $A$ in position $(i,j)$. With $t = 1$, the inverse matrix is given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{-1} \begin{pmatrix} -3 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}^T = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

(c) The linear system has a solution when $|A| = -t \neq 0$, that is when $t \neq 0$. We consider the case $t = 0$: The linear system has infinitely many solutions in thise case since $|A| = 0$ and $\mathbf{b} = \mathbf{0}$, i.e., there is no pivot in the final column and at least one degree of freedom. We conclude that $A\mathbf{x} = \mathbf{b}$ has a least one solution for all values of $t$.

**Question 4.**

(a) The function $f$ is defined for all $(x,y)$ such that $x + y + 1 \neq 0$, that is $x + y \neq -1$. We compute the partial derivatives of $f$ by using the quotient rule for differentiation:

$$f'_x = \frac{yu - xy \cdot 1}{u^2} = \frac{y(x+y+1) - xy}{u^2} = \frac{y(y+1)}{u^2}$$
$$f'_y = \frac{xu - xy \cdot 1}{u^2} = \frac{x(x+y+1) - xy}{u^2} = \frac{x(x+1)}{u^2}$$

We write $u = x + y + 1$ for the denominator in order to make the expressions shorter. The stationary points are given by $f'_x = f'_y = 0$, which gives $y(y+1) = 0$ and $x(x+1) = 0$. Hence, $x = 0$ or $x = -1$, and $y = 0$ or $y = -1$, ad we get the points $(x,y) = (0,0), (-1,0), (0,-1), (-1,-1)$. We see that in these points, $u = 1$ i $(0,0)$, $u = 0$ in $(0,-1)$ and $(-1,0)$, and $u = -1$ in $(-1,-1)$. Hence, the stationary points for $f$ are only the points

$$(x,y) = (0,0), (-1,-1)$$

(b) In order to use the second derivative test, we find the Hessian matrix in the two stationary points. We begin by computing the second order partial derivatives:

$$f''_{xx} = \left(\frac{y(y+1)}{u^2}\right)'_x = y(y+1) \cdot (-2)u^{-3} \cdot 1 = \frac{-2y(y+1)}{u^3}$$
$$f''_{xy} = \left(\frac{y(y+1)}{u^2}\right)'_y = \frac{(2y+1) \cdot u^2 - y(y+1) \cdot 2u \cdot 1}{u^4}$$
$$= \frac{(2y+1)(x+y+1) - 2y(y+1)}{u^3} = \frac{2xy + x + y + 1}{u^3}$$
$$f''_{yy} = \left(\frac{x(x+1)}{u^2}\right)'_y = x(x+1) \cdot (-2)u^{-3} \cdot 1 = \frac{-2x(x+1)}{u^3}$$

We see that $f''_{xx} = f''_{yy} = 0$ for each of the two stationary points since $x(x+1) = y(y+1) = 0$. We use the expression for $f''_{xy}$ above to determine the Hessian in the two stationary points:

$$H(f)(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H(f)(-1,-1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Since the determinant of the two matrices is $-1$, the stationary points of $f$ are saddle points. Since these are the only candidates for the maximum and minimum for $f$, the function $f$ has neither maximum nor minimum value.

**Question 5.**

(a) We use the Lagrange multiplier method with $\mathcal{L} = x - y - \lambda(x^2 + xy + y^2 - 3)$ in order to find candidate points. The Lagrange conditions are

$$\mathcal{L}'_x = 1 - \lambda(2x + y) = 0$$
$$\mathcal{L}'_y = -1 - \lambda(x + 2y) = 0$$
$$x^2 + xy + y^2 = 3$$

We find $x$ and $y$ expressed via $\lambda$ from the first two equations. We get

$$2x + y = 1/\lambda, \quad x + 2y = -1/\lambda$$

In order to simplify the writing, we let $t = 1/\lambda$. Then, we solve the two equations for $x$ and $y$, for instance by Gaussian elimination:

$$\left(\begin{array}{cc|c} 2 & 1 & t \\ 1 & 2 & -t \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & -t \\ 2 & 1 & t \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & -t \\ 0 & -3 & 3t \end{array}\right)$$

By back substitution, this gives $-3y = 3t$, or $y = -t$, and $x + 2(-t) = -t$, or $x = t$. Then, we insert these expressions into the constraint and get that $x^2 + xy + y^2 = t^2 + t(-t) + (-t)^2 = 3$, or $t^2 = 3$, which gives $t = \pm\sqrt{3}$. Since $t = 1/\lambda$, we get $\lambda = 1/t = \pm 1/\sqrt{3}$. This gives the candidate points:

$$(x,y;\lambda) = (\sqrt{3}, -\sqrt{3}; 1/\sqrt{3}),\ (-\sqrt{3}, \sqrt{3}; -1/\sqrt{3})$$

with $f(\sqrt{3}, -\sqrt{3}) = 2\sqrt{3}$ og $f(-\sqrt{3},\sqrt{3}) = -2\sqrt{3}$.

(b) A points has degenerate constraint if $g'_x = g'_y = 0$, where $g(x,y) = x^2 + xy + y^2$. This gives

$$g'_x = 2x + y = 0,\ g'_y = x + 2y = 0$$

This implies that $y = -2x$ from the first equation, and $x + 2(-2x) = 0$, or $-3x = 0$ when inserted into the second equation. Hence, the only point with degenerate constraint is $(x,y) = (0,0)$, but this is not a admissible point since $g(0,0) = 0 \neq 3$. We conclude that there are no admissible points with degenerate constraint for this problem.

(c) Note that the set $D$ of feasible points, given by the equation $g(x,y) = x^2 + xy + y^2 = 3$, is a compact set: It is closed since it is given by an equation (i.e., an equality), and it is also bounded: To see this, we write the equation in the following way by completing the squares:

$$x^2 + xy + y^2 = (x + \frac{1}{2}y)^2 + y^2 - \frac{1}{4}y^2 = (x + \frac{1}{2}y)^2 + \frac{3}{4}y^2 = 3$$

Since the left hand side is a sum of squares, we get that $(x + y/2)^2 \leq 3$ and $3y^2/4 \leq 3$. The final inequality gives $y^2 \leq 4$, i.e., $-2 \leq y \leq 2$. For each $y$-value in this interval, the following must hold: $-\sqrt{3} \leq x + y/2 \leq \sqrt{3}$. Hence, $-\sqrt{3} - y/2 \leq x \leq \sqrt{3} - y/2$. By using the interval of possible $y$-values, we see that $-\sqrt{3} - 1 \leq x \leq \sqrt{3} + 1$. We conclude that the set of feasible points is bounded, and hence compact. By the extreme value theorem, the problem has a maximum, and the only candidates for a maximum are those we found in (a). Hence, we get that

$$f_{\max} = f(\sqrt{3}, -\sqrt{3}) = 2\sqrt{3}$$

since this candidate points has the largest $f$-value of the two points.