Exam	EBA 1180 Mathematics for Data Science
Date	December 18th 2023 at 0900 - 1400

The exam consists of 15 problems with equal weight. All answers must be justified.

Question 1.
Consider the matrix A and the vectors $\mathbf{b}$ and $\mathbf{w}$ given by

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 5 & 7 \\ 1 & 2 & 4 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 14 \\ 10 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Denote the column vectors of A by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

(a) Solve the linear system $A\mathbf{x} = \mathbf{b}$.
(b) Determine all (a,b,c) such that $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

Question 2.
Compute the following integrals:

a) $\displaystyle\int_1^2 4x \ln x \; dx$ b) $\displaystyle\int_0^1 \frac{3x}{\sqrt{x+1}} \; dx$ c) $\displaystyle\int_0^1 \frac{x}{x^2 - 5x + 6} \; dx$ d) $\displaystyle\int e^{\sqrt{x}} \; dx$

Let R be the part of the plane in the first quadrant that is bounded by the graph of $f(x) = x^3 - x$, the straight line L through the origin with slope 3, and the x-axis.

e) Make a figure where R is shown. Compute the area of R.

Question 3.
Let the matrix A and the vector $\mathbf{b}$ be given by

$$A = \begin{pmatrix} t & 1 & t \\ 1 & t & 2 \\ t & 2 & t \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

(a) Compute $|A|$.
(b) Find A^{-1} when $t = 1$.
(c) Determine all values of t such that $A\mathbf{x} = \mathbf{b}$ has at least one solution.

Question 4.
Consider the function f given by $f(x,y) = \frac{xy}{x+y+1}$.

(a) Find the stationary points of the function f.
(b) Classify the stationary points of f. Does f have maximal or minimal values?

Question 5.
Consider the Lagrange problem $\max f(x,y) = x - y$ when $x^2 + xy + y^2 = 3$.

(a) Write down the three Lagrange conditions, and find all points $(x,y;\lambda)$ that satisfy them.
(b) Are there any admissible points with degenerate constraint in this problem?
(c) Determine whether the Lagrange problem has a maximal value. If so, find this value.

Formula Sheet

Financial mathematics

Geometric series.
A finite geometric series has sum

$$S_n = a_1 \cdot \frac{1-k^n}{1-k} = a_1 \cdot \frac{k^n-1}{k-1}$$

and an infinite geometric series has sum

$$S = a_1 \cdot \frac{1}{1-k} \quad \text{when } |k| < 1$$

Present values.
The present value K_0 of a payment K is given by

$$K_0 = \frac{K_n}{(1+r)^n} \quad \text{and} \quad K_0 = \frac{K_n}{e^{rn}}$$

using discrete and continuous compounding.

Integration

Integration techniques.

a) Integration by parts:

$$\int u'v \, \mathrm{d}x = uv - \int uv' \, \mathrm{d}x$$

b) Substitution:

$$\int f(u)u' \, \mathrm{d}x = \int f(u) \, \mathrm{d}u$$

c) Partial fractions:

$$\int \frac{px+q}{(x-a)(x-b)} \, \mathrm{d}x = \int \left(\frac{A}{x-a} + \frac{B}{x-b} \right) \mathrm{d}x$$

Area.
The area of the region bounded by $a \le x \le b$ and $f(x) \le y \le g(x)$ is given by

$$A = \int_a^b (g(x) - f(x)) \, \mathrm{d}x$$

Linear algebra

Cramer's rule.
A linear system $A\mathbf{x} = \mathbf{b}$ where $|A| \neq 0$ has a unique solution given by

$$x_1 = \frac{|A_1(\mathbf{b})|}{|A|} \quad x_2 = \frac{|A_2(\mathbf{b})|}{|A|} \quad \ldots \quad x_n = \frac{|A_n(\mathbf{b})|}{|A|}$$

where $A_i(\mathbf{b})$ is the matrix obtained by replacing column i of A by $\mathbf{b}$.

Functions of two variables

Second derivative test.
A stationary point (x^*, y^*) of the function $f(x,y)$ is a

a) local minimum if $A > 0$ and $AC - B^2 > 0$
b) local maximum if $A < 0$ and $AC - B^2 > 0$
c) saddle point if $AC - B^2 < 0$

when $H(f)(x^*, y^*) = \left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)$.

Level curves.
The slope $y' = \mathrm{d}y/\mathrm{d}x$ of the tangent line to the level curve $f(x,y) = c$ is given by

$$y' = -\frac{f'_x}{f'_y}$$

Method of Lagrange multipliers.
The Lagrange conditions for the problem

$$\max / \min f(x,y) \text{ when } g(x,y) = a$$

is given by

$$\mathcal{L}'_x = 0,\ \mathcal{L}'_y = 0,\ g(x,y) = a$$

An admissible point has degenerated constraint if

$$g'_x = 0,\ g'_y = 0$$