School exam (3h) EBA11805 - Mathematics for Data Science
16 May 2024

SOLUTIONS

Problem 1

The centre of the ellipse is $(3, 2)$ with horizontal semi-axis $a = 3$ and vertical semi-axis $b = 2$. Then, the standard equation for the ellipse is

$$\underline{\underline{\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1}}$$

Problem 2

We see that $(20, 10)$ is the intersection point of the two asymptotes, hence the standard form for the hyperbola function is $f(x) = 10 + \frac{a}{x-20}$ for some number a. We also see that $(21, 5)$ is on the graph of $f(x)$, so $f(21) = 5$, and so $10 + \frac{a}{21-20} = 5$. We solve the equation and get $a = -5$. Hence, $\underline{\underline{f(x) = 10 - \frac{5}{x-20}}}$.

Problem 3

Since P is the maximum point on the graph, the symmetry line for the second degree polynomial function is $x = 6$ og the maximal value is $y = 10$. Hence the standard form is $f(x) = a(x-6)^2 + 10$ for some number a. From $f(4) = 8$ we get the equation $a(4-6)^2 + 10 = 8$ which gives $a = -0.5$. Hence $\underline{\underline{f(x) = -0.5(x-6)^2 + 10}}$.

Problem 4

i) We see that the graph of $f(x)$ is below the graph of $g(x)$ for x between 1.2 and 2 and for x between 4.4 and 7.4. Hence the solution of the inequality is $\underline{\underline{1.2 \leqslant x \leqslant 2 \text{ or } 4.4 \leqslant x \leqslant 7.4}}$.

ii) The product $f(x) \cdot g(x)$ is larger than 0 if both factors have the same sign. This happens when x is between 4 and 5 and when x is between 7 and 8. Hence the solution of the inequality is $\underline{\underline{x \in [4, 5] \text{ or } x \in [7, 8]}}$.

Problem 5

i) If we follow the tangent of the graph of $f(x)$ in $(4, 0)$ backwards to $x = 3$ it seems that the tangent will pass through the point $(3, 0.7)$. Hence, the slope of the tangent is $\underline{\underline{f'(4) \approx -0.7}}$.

ii) We have that $f''(x)$ is negative when the graph bends down (is concave) and $f''(x)$ is positive when the the graph bends upwards (is convex). The boundary between convex/concave are the inflection points and they seem to be $x = 4$ and $x = 7.5$. Hence,
$\underline{\underline{f''(x) \leqslant 0 \text{ for } x \leqslant 4,\ f''(x) \geqslant 0 \text{ for } 4 \leqslant x \leqslant 7.5 \text{ and } f''(x) \leqslant 0 \text{ for } x \geqslant 7.5}}$.

Problem 6

i) We use the chain rule with $u(x) = x(10-x) = 10x - x^2$ and $g(u) = 30e^u$. Because $u'(x) = 10 - 2x = 2(5-x)$ and $g'(u) = 30e^u$ we get $\underline{\underline{f'(x) = 60(5-x)e^{x(10-x)}}}$. Stationary points for $f(x)$ are zeros for $f'(x)$. So the only stationary point for $f(x)$ is $\underline{\underline{x = 5}}$.

ii) Maximum/minimum for $f(x)$ is either at a stationary point or in an end point. Because $f(3) = 30e^{21}$, $f(5) = 30e^{25}$ and $f(8) = 30e^{16}$ we get the minimal value is $f(8) = 30e^{16}$ and the maximal value is $f(5) = 30e^{25}$.

Problem 7

i) The monthly interest is $6\%/12 = 0.5\%$. The first payment is in the account for $12 \cdot (12-4) = 96$ interest periods. There are 97 payments. Hence, the geometric series for the balance 12 years from now is

$$15\,000 \cdot 1.005^{96} + 15\,000 \cdot 1.005^{95} + \cdots + 15\,000 \cdot 1.005 + 15\,000$$

ii) If we read the geometric series backwards, we get the first term $a_1 = 15\,000$, the multiplication factor $k = 1.005$ and the number of terms $n = 97$. The formula for a geometric series then gives

$$15\,000 \cdot \frac{1.005^{97} - 1}{0.005} = 1\,866\,640.27$$

Problem 8

i) Let r denote the IRR. Then the present value of the cash flow is equal to 0. This gives the equation

$$-20 - \frac{20}{(1+r)} + \frac{25}{(1+r)^3} + \frac{40}{(1+r)^5} = 0$$

ii) If we put $r = 14\%$ into the left hand side, we get that the present value is 0.1052. If r increases, the powers in the denominators will also increase and the values of the positive fractions will decrease (the present values of future payments decreases when the interest increases). The value of the negative fraction ($-\frac{20}{(1+r)}$) will increase (become less negative), but this effect is smaller than the effect of the positive fractions becoming smaller. We can try, say with $r = 14.05\%$. Then the present value is 0.0452. Hence the IRR is (a little) larger than 14.05%.

Problem 9

i) We see that $f(x)$ is defined for all x, hence $f(x)$ has no vertical asymptotes. But

$$f(x) = 4 + 5e^{-0.1x} = 4 + \frac{5}{e^{0.1x}} \underset{x\to\infty}{\longrightarrow} 4 + 0^+ = 4^+.$$

Hence $y = 4$ is a horizontal asymptote.

ii) To find the expression for $g(x)$ we put $y = f(x)$ and solve for x: Subtract 4 from both sides and get $y - 4 = \frac{5}{e^{0.1x}}$. Multiply both sides with $e^{0.1x}$ and get $(y-4)e^{0.1x} = 5$. Divide by $(y-4)$ on both sides and get $e^{0.1x} = \frac{5}{(y-4)}$. Substitute both sides for x in the function $\ln(x)$. Because $\ln(x)$ and e^x are inverse functions we get $0.1x = \ln \frac{5}{(y-4)}$. Multiplication with 10 on both sides gives $x = 10 \ln \frac{5}{(y-4)}$. Interchange variables and get $g(x) = 10 \ln \frac{5}{(x-4)}$. The domain of definition for $g(x)$ equals the range of $f(x)$. Because $f(x)$ is decreasing, the maximal value is $f(0) = 4 + 5 \cdot 1 = 9$. With the result in (i), $D_g = V_f = \langle 4, 9]$. Moreover, $V_g = D_f = [0, \rightarrow\rangle$.

Problem 10

i) We substitute both sides for x in the function e^x. Since $\ln(x)$ and e^x are inverse functions we get $x^4 - x^2 - 5 = e^0$, that is $x^4 - x^2 - 6 = 0$. Substitute $u = x^2$ and we get the second degree polynomial equation $u^2 - u - 6 = 0$ which has solutions $u = 3$ and $u = -2$. When we substitute back we get the equations $x^2 = 3$ og $x^2 = -2$. The first equation has solutions $\underline{\underline{x = \pm\sqrt{3}}}$, while the second has no solutions.

ii) There are many ways to choose a parameter. We can f. ex. choose $x = s$ to be the symmetry line for the quadratic function (hence the point in the middle between the roots). The smallest root (zero) is then $x = s - 3$ while the larger root is $x = s + 3$. Hence, we can write

$$x^2 + bx + c = [x-(s-3)][x-(s+3)] = x^2 - (s+3)x - (s-3)x + (s-3)(s+3) = \underline{\underline{x^2 - 2sx + s^2 - 9}}$$

In particular $\underline{\underline{b = -2s}}$ og $\underline{\underline{c = s^2 - 9}}$.

Problem 11

i) We have $P_3(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2}\cdot x^2 + \frac{f'''(0)}{6}\cdot x^3$. By applying the chain rule with $u = x+1$ we get $f'(x) = \frac{1}{x+1}$, $f''(x) = \frac{-1}{(x+1)^2}$ and $f'''(x) = \frac{2}{(x+1)^3}$. We get $f(0) = \ln(0+1) = 0$, $f'(0) = \frac{1}{0+1} = 1$, $f''(0) = \frac{-1}{(0+1)^2} = \frac{-1}{1} = -1$ and $f'''(0) = \frac{2}{(0+1)^3} = 2$. Hence $\underline{\underline{P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}}}$.

ii) $\ln(1.2) = \ln(0.2+1) \approx P_3(0.2) = 0.2 - \frac{0.2^2}{2} + \frac{0.2^3}{3} = 0.2 - 0.02 + 0.002667 = \underline{\underline{0.1827}}$

Problem 12

i) If r_{eff} is the effective annual interest, $K_{10} = K_0 \cdot (1 + r_{\text{eff}})^{10}$. We solve this equation for r_{eff}. Dividing by K_0 on both sides:

$$(1 + r_{\text{eff}})^{10} = \frac{K_{10}}{K_0} \quad \text{and raise both sides to the power of } \frac{1}{10}: \quad 1 + r_{\text{eff}} = \left(\frac{K_{10}}{K_0}\right)^{\frac{1}{10}}$$

That is,

$$\underline{\underline{r_{\text{eff}} = \left(\frac{K_{10}}{K_0}\right)^{\frac{1}{10}} - 1}}$$

ii) With continuous compounding the annual growth factor is e^r when r is the nominal annual interest. Because the annual growth factor also equals $1 + r_{\text{eff}}$ we get

$$e^r = 1 + r_{\text{eff}} = \left(\frac{K_{10}}{K_0}\right)^{\frac{1}{10}}$$

Insert into $\ln(x)$ and get

$$\underline{\underline{r = \ln(1 + r_{\text{eff}}) = \frac{1}{10}(\ln K_{10} - \ln K_0)}}$$