School exam (3h) EBA11805 - Mathematics for Data Science

6 December 2024

SOLUTIONS

Problem 1

i) We use the chain rule with $u = u(x) = 0.5x^2 - 3x$ as inner function which gives $g(u) = e^u$ as outer function. Then $f'(x) = u'(x) \cdot g'(u) = \underline{\underline{(x-3)e^{0.5x^2-3x}}}$.

ii) The stationary points of $f(x)$ are the solutions of the equation $f'(x) = 0$, that is $(x-3)e^{0.5x^2-3x} = 0$. Because $e^{0.5x^2-3x} > 0$, the only possibility is $\underline{\underline{x = 3}}$. To determine where $f(x)$ is increasing and decreasing we look at the sign of $f'(x)$ which is negative if $x < 3$ and positive if $x > 3$. hence $\underline{\underline{f(x) \text{ decreasing for } x \in \langle \leftarrow, 3]}}$ and $\underline{\underline{\text{increasing for } x \in [3, \rightarrow\rangle}}$.

Problem 2

We divide each side of the equation with 36 to get it on standard form:

$$\frac{x^2}{6^2} + \frac{(y-5)^2}{2^2} = 1$$

Then the centre of the ellipse is $(0, 5)$, the horizontal semi-axis is $a = 6$ and the vertical semi-axis is $b = 2$.

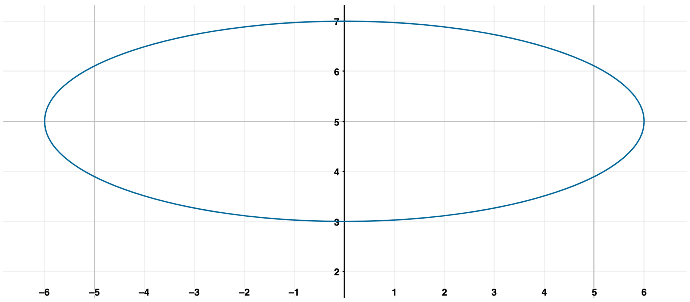

Figure 1: Ellipse

Problem 3

We draw the line through the origin A which is tangent to the graph of the cost function, see figure 2. Then the x-coordinate of the tangent point B is the cost optimum and the slope of the line is minimal average unit cost. On the figure we see $B \approx (20, 109)$ which gives $\underline{\underline{\text{cost optimum} \approx 20}}$ and $\underline{\underline{\text{minimal average unit cost} \approx \frac{109}{20} = 5.45}}$.

Problem 4

i) We read the geometric series backwards such that the first term is $a_1 = 8\,000 \cdot 1.005^{24}$, the multiplication factor is $k = 1.005$, and the number of terms is $n = 143 - 23 = 120$. Then the formula for the sum of a geometric series gives

$$8\,000 \cdot 1.005^{143} + 8\,000 \cdot 1.005^{142} + 8\,000 \cdot 1.005^{141} + \ldots + 8\,000 \cdot 1.005^{25} + 8\,000 \cdot 1.005^{24}$$

$$= 8\,000 \cdot 1.005^{24} \cdot \frac{1.005^{120} - 1}{0.005} = \underline{\underline{1\,477\,745.66}}.$$

ii) If we read the series from the left it can be the balance (future value) of a bank account in 12 years from now with 6% nominell interest, monthly compounding (with monthly interest $6\%/12 = 0.005$), deposits of 8 000 each month, first deposit in one month from now, the last 10 years from now (so 120 deposits) which remain in the account for two years after last deposit.

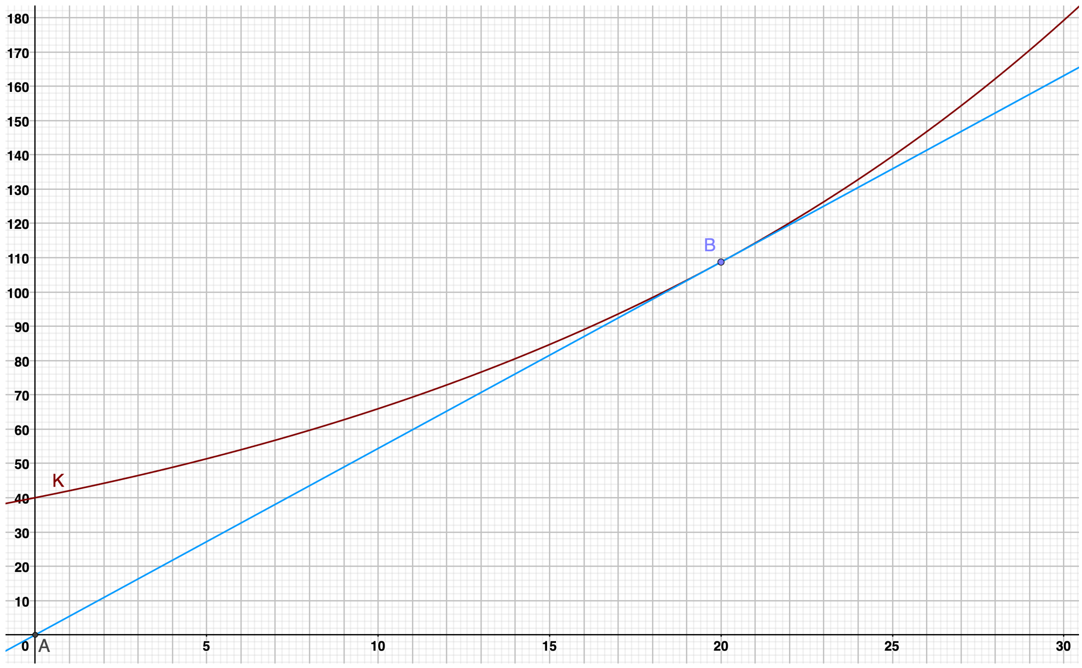

Figure 2: Cost function

Problem 5

We see that $(30\,,50)$ is the symmetry point of the hyperbola (the line through the points $(26\,,47)$ and $(33\,,51)$, and the line through the points $(28\,,48)$ and $(32\,,52)$ intersect at $(30\,,50)$). Hence we have two of the three numbers in the standard form of a hyperbola function: $f(x)=50+\frac{a}{x-30}$. To determine a, we insert the point $(32\,,52)$ in $f(x)$, that is $f(32)=52$ which gives the equation $50+\frac{a}{32-30}=52$. We solve it and get $a=4$. Hence $\underline{\underline{f(x)=50+\frac{4}{x-30}}}$.

Problem 6

In the standard form for a second degree polynomial function $f(x)=a(x-s)^2+d$ the maximum point gives $s=12$ and $d=24$, that is $f(x)=a(x-12)^2+24$. Inserting P gives the equation $a(10-12)^2+24=20$, that is $a=-1$ and $\underline{\underline{f(x)=-(x-12)^2+24}}$.

Problem 7

The function has to be strictly concave for x between 5 and 15, and strictly convex for $x \geqslant 15$. Then the graph can look like figure 3.

Problem 8

i) We have $D'(p)=-1/p=-p^{-1}$ so $\varepsilon(p)=\frac{-p^{-1}p}{-\ln(p)}=\underline{\underline{\frac{1}{\ln(p)}}}$.

ii) The demand is elastic if $\varepsilon(p)<-1$, that is $\frac{1}{\ln(p)}<-1$, that is $\frac{\ln(p)+1}{\ln(p)}<0$. We have $\ln(p)<0$ for $0<p<1$ hence the inequality is $\ln(p)+1>0$, that is $\ln(p)>-1$, which gives $\underline{\underline{p\in\langle e^{-1},1\rangle}}$.

iii) Because $p=0.5>e^{-1}$, the demand will be elastic and a small price increase from 0.5 will give a lower income.

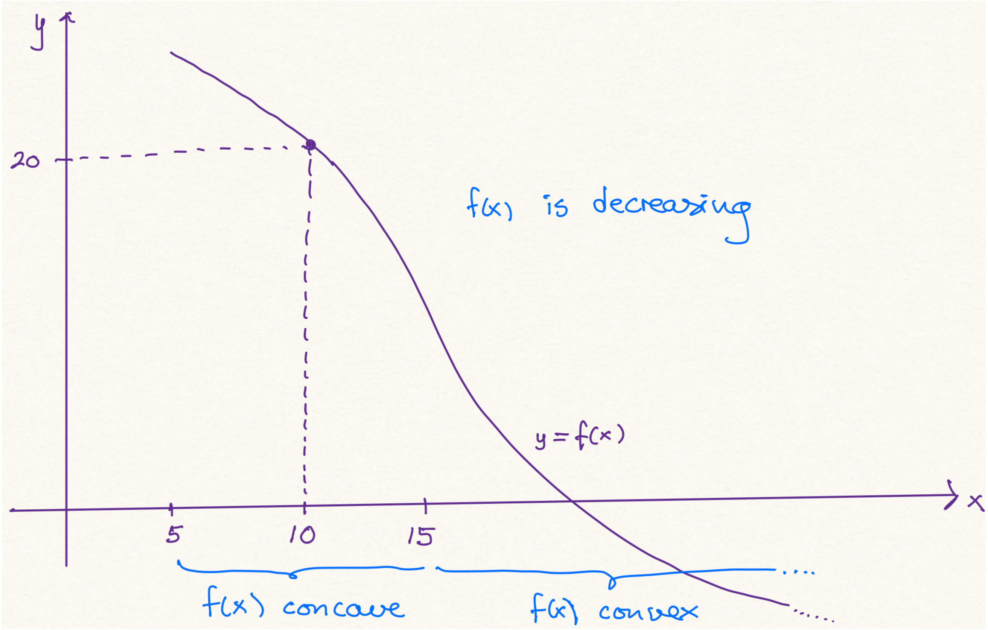

Figure 3: Decreasing graph

Problem 9

To answer the questions we make a sign diagram for $f''(x)$. To do so we need the zeros of $f''(x)$, that is the solutions of the equations $e^x - 7 = 0$ and $\ln(x^2 + 0.5) = 0$. For $e^x - 7 = 0$ we get $e^x = 7$, that is $x = \ln(7)$. For $\ln(x^2 + 0.5) = 0$ we get $x^2 + 0.5 = 1$, that is $x^2 = 0.5$, which gives $x = \pm\frac{1}{\sqrt{2}}$. Then we get a sign diagram as in figure 4:

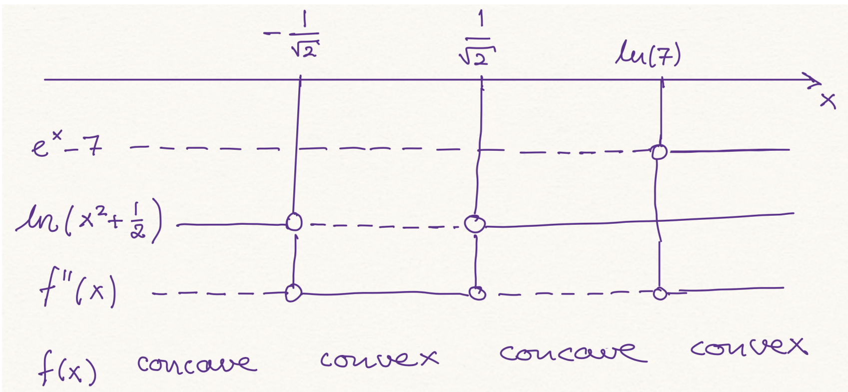

Figure 4: Sign diagram

Hence we get:

i) The inflection points of $f(x)$ are $\underline{\underline{x = \pm\frac{1}{\sqrt{2}}}}$ and $\underline{\underline{x = \ln(7)}}$.

ii) $f(x)$ is concave in the interval $\langle\leftarrow, -\frac{1}{\sqrt{2}}]$, convex in the interval $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, concave in the interval $[\frac{1}{\sqrt{2}}, \ln(7)]$ and convex in the interval $[\ln(7), \rightarrow\rangle$.

Problem 10

i) The present value of the cash flow is the sum of the present values of the payments:

$$\underline{\underline{\frac{A}{e^{3r}} + \frac{B}{e^{4r}} + \frac{C}{e^{5r}}}}$$

ii) We have to solve the inequality $\frac{100}{e^{5r}} > \frac{60}{e^{4r}}$. We multiply both sides with $\frac{e^{5r}}{60} > 0$ and get $\frac{100}{60} > e^r$, that is $e^r < \frac{5}{3}$, which gives $\underline{\underline{r < \ln(5) - \ln(3) = 51.08\%}}$.

Problem 11

i) We solve the equation $y = \frac{60e^{-0.01x}+100}{e^{-0.01x}+4}$ for x. First we multiply with the denominator $e^{-0.01x} + 4$ on each side and get

$$e^{-0.01x}y + 4y = 60e^{-0.01x} + 100$$

Subtracting 100 and $e^{-0.01x}y$ on each side and factorising the right hand side gives

$$4y - 100 = (60 - y)e^{-0.01x}$$

Dividing with $60 - y$ on each side gives

$$\frac{4y - 100}{60 - y} = e^{-0.01x}$$

Inserting both sides into $\ln(-)$ and multiplying with -100 on each side gives

$$-100 \ln \frac{4y - 100}{60 - y} = x$$

Exchanging variables, we get an expression for the inverse function

$$\underline{\underline{g(x) = -100 \ln \frac{4x - 100}{60 - x}}}$$

ii) The domain D_g equals the range R_f. We have $f(0) = \frac{60+100}{1+4} = 32$. Moreover

$$\begin{aligned} f'(x) &= \frac{-0.6e^{-0.01x}(e^{-0.01x} + 4) - (60e^{-0.01x} + 100)(-0.01e^{-0.01x})}{(e^{-0.01x} + 4)^2} \\ &= \frac{-0.6e^{-0.02x} - 2.4e^{-0.01x} + 0.6e^{-0.02x} + e^{-0.01x}}{(e^{-0.01x} + 4)^2} = \frac{-1.4e^{-0.01x}}{(e^{-0.01x} + 4)^2} < 0 \end{aligned}$$

for all x. Hence $f(x)$ is decreasing. We have $f(x) \underset{x\to\infty}{\longrightarrow} \frac{60\cdot 0+100}{0+4} = 25$ and so $D_g = R_f = \underline{\underline{\langle 25, 32]}}$ and the range $R_g = D_f = \underline{\underline{[0, \rightarrow\rangle}}$.

Problem 12

By polynomial division we get that $f(x)$ is

$$\begin{array}{l} \big(x^3 + 7x^2 - 100x - 700\big) : \big(x^2 + 2x - 35\big) = x + 5 + \dfrac{-75x - 525}{x^2 + 2x - 35} \\ \underline{-x^3 - 2x^2 \;\; + 35x} \\ \qquad\quad 5x^2 \;\; - 65x - 700 \\ \qquad \underline{-5x^2 \;\; - 10x + 175} \\ \qquad\qquad\quad -75x - 525 \end{array}$$

We can simplify the last fraction because $x^2 + 2x - 35 = (x + 7)(x - 5)$ and $-75x - 525 = -75(x + 7)$. Hence we get $f(x) = x + 5 - \frac{75}{x-5}$. We calculate $f'(x) = 1 + \frac{75}{(x-5)^2}$ and see that $f'(x) > 1$. In particular $f(x)$ is strictly increasing for $x > 5$. Hence $f(x)$ has neither any maximum nor any minimum.