School exam (3h) EBA11805 - Mathematics for Data Science

5 December 2025

SOLUTIONS

Problem 1

i) We calculate:

$$\begin{array}{l}
(\quad x^3+11x^2+40x+50):(x+5)=x^2+6x+10\\
\underline{-x^3\ \ -5x^2}\\
\qquad\quad 6x^2+40x\\
\qquad \underline{-6x^2-30x}\\
\qquad\qquad\quad 10x+50\\
\qquad\qquad \underline{-10x-50}\\
\qquad\qquad\qquad\quad 0
\end{array}$$

The remainder is 0.

ii) Because $x^2+6x+10=(x+3)^2+1$ does not have any zeros, the expression cannot be factorised any further. Hence $x^3+11x^2+40x+50=\underline{\underline{(x+5)(x^2+6x+10)}}$.

Problem 2

i) Read the geometric series backwards and get the first term $a_1=\frac{20\,000}{1.03^{25}}$, the multiplication factor $k=1.03$ and number of terms $n=25-1=24$. Put into the formula for the sum of a geometric series we get:

$$\frac{20\,000}{1.03^{25}}\cdot\frac{1.03^{24}-1}{0.03}=\underline{\underline{328\,845.48}}$$

ii) We have a regular cash flow with 20 000 payed every year, the first payment 2 years from now and the last payment 25 years from now. The sum of the present value of the payments is the present value of this cash flow if the interest is $r=3\%$ (with the first payment to the left in the sum). The present value can then be the loan amount one will obtain if one pays 20 000 every year as described.

Problem 3

i) $\underline{\underline{\text{FALSE}}}$. The differentiated function gives the slopes of the tangents of $f(x)$. It seems that $f'(3)\approx 0$ while $f'(8)\approx 0.4$. Hence $f'(3)<f'(8)$.

ii) $\underline{\underline{\text{FALSE}}}$. $x=a$ is a stationary point if $f'(a)=0$. In the interval there are two such points: $x\approx 3$ and $x\approx 5.3$.

iii) $\underline{\underline{\text{TRUE}}}$. $x=a$ is an inflection point if $f''(x)$ changes sign at $x=a$. This happens where the graph changes form from convex to concave or opposite. In the interval this happens for $x\approx 3.7$ (from concave to convex) and for $x\approx 6.8$ (from convex to concave).

Problem 4

The standard form for a hyperbola function is $f(x)=c+\frac{a}{x-b}$. Here $x=b$ is a vertical asymptote, so $b=30$, and $y=c$ is a horizontal asymptote, so $c=50$. This gives $f(x)=50+\frac{a}{x-30}$. From $f(33)=48$ we get $50+\frac{a}{33-30}=48$ which gives $a=-6$. Hence $\underline{\underline{f(x)=50-\frac{6}{x-30}}}$.

Problem 5

i) Add $g(x)$ to each side and get the equivalent inequality $f(x) > g(x)$. This happens where the graph of $f(x)$ is above $g(x)$. This gives the (approximate) solutions $\underline{\underline{2 < x < 4.37}}$ og $\underline{\underline{x > 7.4}}$.

ii) Here we make a sign diagram:

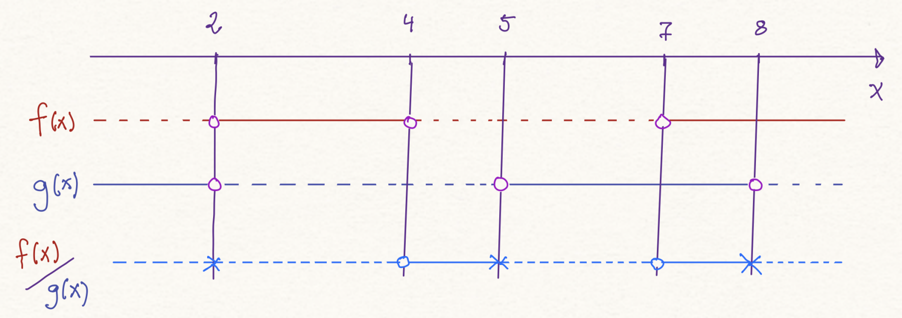

Figure 1: Sign diagram

Hence the (approximate) solutions are $\underline{\underline{x < 2}}$ and $\underline{\underline{2 < x \leqslant 4}}$ and $\underline{\underline{5 < x \leqslant 7}}$ and $\underline{\underline{x > 8}}$.

Problem 6

i) Annual growth factor is 1.046. The growth factor for 12 years is 1.046^{12}. The deposit is the present value $\frac{800\,000}{1.046^{12}} = \underline{\underline{466\,347.63}}$.

ii) Let r be the interest. Then $(1+r)^6$ is the growth factor for 6 years. Hence we have the equation $\frac{800\,000}{1.046^{12}} \cdot (1+r)^6 = 800\,000$. This gives the equation $(1+r)^6 = 1.046^{12}$, which simplifies to $1+r = 1.046^2$, so that $r = 1.046^2 - 1 = \underline{\underline{9.41\%}}$.

Problem 7

i) We have the standard equation $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$ where $(x_0, y_0) = (3, 5)$, $a = 2$ and $b = 4$. This gives

$$\underline{\underline{\frac{(x-3)^2}{4} + \frac{(y-5)^2}{16} = 1}}$$

ii)

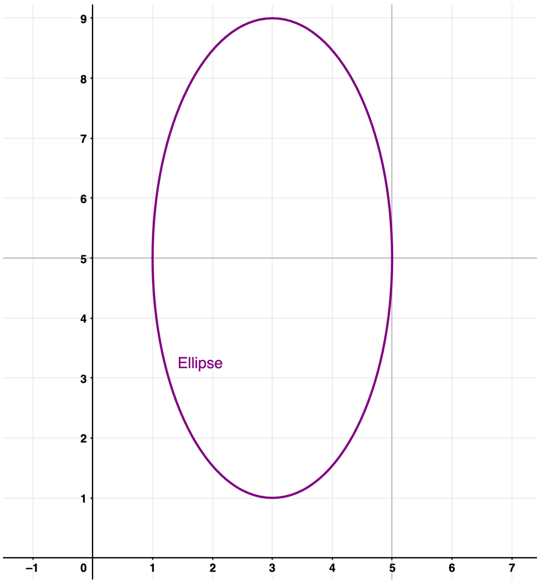

Figure 2: Ellipse

Problem 8

The function has to be decreasing since $f'(x) < 0$, convex for $x < 15$ because $f''(x) > 0$ and concave for $x > 15$ because $f''(x) < 0$.

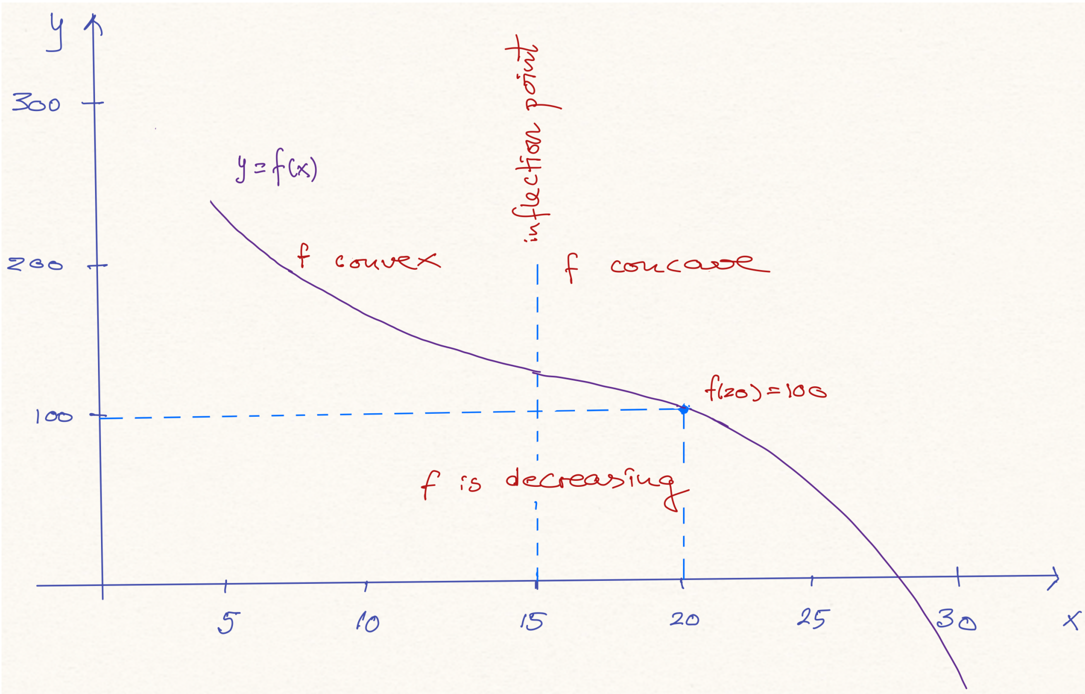

Figure 3: The graph of $f(x)$ – one of many possibilities

Problem 9

i) By the chain rule we have $D'(p) = 150(-0.01 \cdot 2p)e^{-0.01p^2}$. Hence:

$$\varepsilon(p) = \frac{D'(p) \cdot p}{D(p)} = \frac{(-0.02p)150e^{-0.01p^2}p}{150e^{-0.01p^2}} = \underline{\underline{-0.02p^2}}$$

ii) We get elastic demand when $\varepsilon(p) < -1$, that is $-0.02p^2 < -1$, which simplifies to $p^2 > 50$. Because the price is non-negative, we get $\underline{\underline{p > \sqrt{50}}}$.

iii) Because $10 > \sqrt{50}$, the demand vil be price elastic for $p = 10$. That implies that the revenue decreases if the price increases a little from 10.

Problem 10

We consider y as a function of x and differentiate each side of the equation w.r.t. x by applying the chain rule on the left hand side and the product rule on the right hand side:

$$e^y y' = xy' + y \quad \text{which gives} \quad (e^y - x)y' = y \quad \text{and} \quad \underline{\underline{y' = \frac{y}{e^y - x}}}$$

Problem 11

We see that $e^{0.03x} \xrightarrow[x\to-\infty]{} 0$ such that $\frac{50e^{0.03x}+1}{120e^{0.03x}+2025} \xrightarrow[x\to-\infty]{} \frac{1}{2025}$. Thus

$$f(x) \xrightarrow[x\to-\infty]{} \ln\left(\frac{1}{2025}\right) = -\ln(2025)$$

and $\underline{\underline{y = -\ln(2025) \text{ is a horizontal asymptote for } f(x)}}$.

We divide the numerator and the denominator in the expression $\frac{50e^{0.03x}+1}{120e^{0.03x}+2025}$ with $e^{0.03x}$, and get $\frac{50+e^{-0.03x}}{120+2025e^{-0.03x}}$. Thus

$$f(x) \xrightarrow[x\to\infty]{} \ln\left(\frac{50}{120}\right) = \ln(5) - \ln(12)$$

and $\underline{\underline{y = \ln(5) - \ln(12) \text{ is a horizontal asymptote for } f(x)}}$.

For $f(x)$ to have a vertical asymptote, $\frac{50e^{0.03x}+1}{120e^{0.03x}+2025} = 0$ must have an solution $x = a$, but the numerator is greater than 1 for all x so this is not possible. Hence $\underline{\underline{f(x) \text{ has no vertical asymptote}}}$.

Problem 12

We have $P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$. We therefore have to differentiate $f(x)$ three times. We use the product rule and the chain rule with $u = x+1$ and get:

$$f'(x) = e^x \ln(x+1) + \frac{e^x}{x+1}$$

The first term is $f(x)$ and for the second term we use the fraction rule:

$$f''(x) = e^x \ln(x+1) + \frac{e^x}{x+1} + \frac{e^x(x+1) - e^x}{(x+1)^2} = e^x \ln(x+1) + \frac{e^x}{x+1} + \frac{xe^x}{(x+1)^2}$$

The first two terms is $f'(x)$. By the fraction rule (and the product rule), differentiation of the last term gives:

$$\left(\frac{xe^x}{(x+1)^2}\right)' = \frac{(x+1)e^x(x+1)^2 - 2x(x+1)e^x}{(x+1)^4} = \frac{((x+1)^2 - 2x)e^x}{(x+1)^3} = \frac{(x^2+1)e^x}{(x+1)^3}$$

Hence we get:

$$f'''(x) = e^x \ln(x+1) + \frac{e^x}{x+1} + \frac{xe^x}{(x+1)^2} + \frac{(x^2+1)e^x}{(x+1)^3}$$

With $x = 0$ we get $f(0) = e^0 \ln(0+1) = 0$, $f'(0) = f(0) + \frac{e^0}{0+1} = 1$, $f''(0) = f'(0) + \frac{0e^0}{(0+1)^2} = 1$ and $f'''(0) = f''(0) + \frac{(0^2+1)e^0}{(0+1)^3} = 2$. Hence $\underline{\underline{P_3(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3}}$.