Solution	EBA 1180 Mathematics for Data Science
Date	December 18th 2025 from 0900 - 1400

Problem 1.

(a) We write the extended matrix $(A|\mathbf{0})$ of the linear system and do elementary row operations until we find an echelon form:

$$\left(\begin{array}{ccccc|c} 1 & 3 & 1 & 2 & 4 & 0 \\ 3 & 4 & 8 & 4 & 5 & 0 \\ 2 & 6 & 2 & 5 & 4 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 3 & 1 & 2 & 4 & 0 \\ 0 & -5 & 5 & -2 & -7 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \end{array}\right)$$

Since the final column is a zero-column and we have pivot positions in columns 1, 2 and 4, we see that x_3 and x_5 are free, and hence there are two degrees of freedom and infinitely many solutions. We use the parameters $x_3 = s$ and $x_5 = t$, as well as back substitution: From the third equation, we get $x_4 - 4t = 0$, or $x_4 = 4t$. From the second equation, we find $-5x_2+5s-2(4t)-7t = 0$, or $x_2 = s-3t$. The first equation gives $x_1+3(s-3t)+s+2(4t)+4t = 0$, or $x_1 = -4s - 3t$. Hence, the solutions of the linear system can be written

$$\mathbf{x} = \begin{pmatrix} -4s-3t \\ s-3t \\ s \\ 4t \\ t \end{pmatrix} = s\begin{pmatrix} -4 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -3 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

where s,t are parameters corresponding to free variables.

(b) To solve the linear system $A\mathbf{x} = \mathbf{b}$, we look at the echelon form of the extended matrix $(A|\mathbf{b})$: The first five columns are unaltered, and for all three rows we have a pivot position within the first five columns. Hence, there are still infinitely many solutions and two degrees of freedom for all vectors $\mathbf{b}$.

Problem 2.

(a) We write $\sqrt{x}/x^2 = x^{1/2}/x^2 = x^{-3/2}$ and use the power rule:

$$\int_1^4 \frac{\sqrt{x}}{x^2}\,\mathrm{d}x = \int_1^4 x^{-3/2}\,\mathrm{d}x = \left[\frac{x^{-1/2}}{(-1/2)}\right]_1^4 = \left[\frac{-2}{\sqrt{x}}\right]_1^4 = -\frac{2}{2} + \frac{2}{1} = 2 - 1 = 1$$

(b) The factorisation $x^2 - 5x - 6 = (x-6)(x+1)$ of the denominator can be used to find a partial fraction decomposition:

$$\frac{7}{x^2-5x+6} = \frac{A}{x-6} + \frac{B}{x+1} \quad \Rightarrow \quad 7 = A(x+1) + B(x-6)$$

This gives $(A+B)x + (A-6B) = 7$, and hence $A + B = 0$ and $A - 6B = 7$. We find that $B = -A$ (from the first equation), and hence we get $A - 6(-A) = 7A = 7$, that is $A = 1$, $B = -1$. This gives the integral

$$\begin{aligned}\int_0^5 \frac{7}{x^2-5x-6}\,\mathrm{d}x &= \int_0^5 \frac{1}{x-6} + \frac{-1}{x+1}\,\mathrm{d}x = \big[\ln|x-6| - \ln|x+1|\big]_0^5 \\ &= (\ln 1 - \ln 6) - (\ln 6 - \ln 1) = -2\ln 6 \approx -3.58\end{aligned}$$

(c) We use partial fractions with $u' = \frac{1}{2\sqrt{x}}$ and $v = \ln x$, and hence $u = \sqrt{x}$ and $v' = 1/x$. The indefinite integral becomes

$$\int \frac{\ln x}{2\sqrt{x}}\,\mathrm{d}x = \sqrt{x}\ln x - \int \sqrt{x}/x\,\mathrm{d}x = \sqrt{x}\ln x - \frac{x^{1/2}}{1/2} + C = \sqrt{x}\ln x - 2\sqrt{x} + C$$

since $\sqrt{x}/x = x^{1/2}/x = x^{-1/2}$. Hence, the definite integral has the value

$$\int_1^4 \frac{\ln x}{2\sqrt{x}}\,\mathrm{d}x = \big[\sqrt{x}\ln x - 2\sqrt{x}\big]_1^4 = (2\ln 4 - 4) - (\ln 1 - 2) = 2\ln 4 - 2 \approx 0.77$$

(d) We use the substitution $u = e^x + 1$, with $\mathrm{d}u = e^x\,\mathrm{d}x$, and get that the indefinite integral becomes

$$\int \frac{1}{e^x+1}\,\mathrm{d}x = \int \frac{1}{u}\cdot\frac{\mathrm{d}u}{e^x} = \int \frac{1}{u(u-1)}\,\mathrm{d}u$$

since $u = e^x + 1$ gives $e^x = u - 1$. The factorisation of the denominator can be used to find a partial fraction decomposition:

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \quad \Rightarrow \quad 1 = A(u-1) + B(u) = (A+B)u - A$$

This gives $A + B = 0$ and $-A = 1$, or $A = -1$ and $B = 1$. Hence, the integral becomes

$$\begin{aligned}\int \frac{1}{e^x+1}\,\mathrm{d}x &= \int \frac{1}{u(u-1)}\,\mathrm{d}u = \int -\frac{1}{u} + \frac{1}{u-1}\,\mathrm{d}u = -\ln|u| + \ln|u-1| + C \\ &= \ln|u-1| - \ln|u| + C = \ln(e^x) - \ln(e^x+1) + C = x - \ln(e^x+1) + C\end{aligned}$$

Problem 3.

(a) Since the payoff per stock in each of the scenarios is given by the difference between the price and the purchase price, the payoffs are given by the table below. Hence, we get that the total payoffs in each scenario are $R_1 = -20x + 40y - 100z$, $R_2 = 50x - 60y + 150z$, $R_3 = 30x + 20y + 50z$.

	A	B	C
Scenario 1	−20	40	−100
Scenario 2	50	−60	150
Scenario 3	30	20	50

(b) We solve the system via Gaussian elimination:

$$\left(\begin{array}{ccc|c} -20 & 40 & -100 & R_1 \\ 50 & -60 & 150 & R_2 \\ 30 & 20 & 50 & R_3 \end{array}\right) \to \left(\begin{array}{ccc|c} -20 & 40 & -100 & R_1 \\ 0 & 40 & -100 & R_2 + 2.5R_1 \\ 0 & 80 & -100 & R_3 + 1.5R_1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} -20 & 40 & -100 & R_1 \\ 0 & 40 & -100 & R_2 + 2.5R_1 \\ 0 & 0 & 100 & R_3 - 2R_2 - 3.5R_1 \end{array}\right)$$

If we insert $R_1 = 20.000$ NOK, $R_2 = 10.000$ NOK, and $R_3 = 70.000$ NOK, we find the echelon form

$$\left(\begin{array}{ccc|c} -20 & 40 & -100 & 20.000 \\ 0 & 40 & -100 & 60.000 \\ 0 & 0 & 100 & -20.000 \end{array}\right)$$

Back substitution gives $100z = -20.000$, or $z = -200$, $40y + 20.000 = 60.000$, or $y = 40.000/40 = 1000$, and $-20x + 40.000 + 20.000 = 20.000$, or $-20x = -40.000$ which gives $x = 2.000$. Hence, it is possible if we purchase 2.000 stocks in firm A, 1.000 stocks in firm B, and −200 stocks (i.e., short selling 200 stocks) in firm C. The initial purchase price (cost price) of this portfolio of stocks is 100.000 NOK since

$$2.000 \cdot 30 + 1.000 \cdot 80 + (-200) \cdot 200 = 100.000$$

Problem 4.

(a) We find the partial derivatives of the function $f(x,y) = xy^3 - x^3y + 4x^2y - 4xy$ by using the rules of differentiation. This gives

$$f'_x = y^3 - 3x^2y + 8xy - 4y$$
$$f'_y = 3xy^2 - x^3 + 4x^2 - 4x$$

We find the second order derivatives and the Hessian matrix of f by partial differentiating the above expressions one more time:

$$H(f) = \begin{pmatrix} -6xy + 8y & 3y^2 - 3x^2 + 8x - 4 \\ 3y^2 - 3x^2 + 8x - 4 & 6xy \end{pmatrix}$$

(b) We solve the first order conditions $f'_x = f'_y = 0$ in order to find the stationary points of f. We see that y is a common factor in the first equation and x is a common factor in the second equation. This gives:

$$f'_x = y(y^2 - 3x^2 + 8x - 4) = 0$$
$$f'_y = x(3y^2 - x^2 + 4x - 4) = 0$$

Since we are to find stationary points where $x,y \neq 0$, we can divide by y and x in the two equations. This gives $y^2 - 3x^2 + 8x - 4 = 0$ in the first equation or $y^2 = 3x^2 - 8x + 4$. We insert this into the second equation, which gives

$$3(3x^2 - 8x + 4) - x^2 + 4x - 4 = 8x^2 - 20x + 8 = 0 \quad \Rightarrow \quad x = \frac{20 \pm \sqrt{400 - 4(8)(8)}}{2(8)}$$

Hence, $x = (20 + 12)/16 = 2$ or $x = (20 - 12)/16 = 1/2$. When $x = 2$ we get $y^2 = 0$, and hence $y = 0$. When $x = 1/2$ we get $y^2 = 3/4$, and hence $y = \pm\sqrt{3}/2$. Note that only the final two points have $x,y \neq 0$, and we therefore get the following stationary points with $x,y \neq 0$:

$$(x,y) = (1/2, \pm\sqrt{3}/2)$$

(c) We use the second derivative test and first insert $(x,y) = (1/2, \sqrt{3}/2)$ in the Hessian matrix from (a):

$$H(f)(1/2,\sqrt{3}/2) = \begin{pmatrix} 5\sqrt{3}/2 & 3/2 \\ 3/2 & 3\sqrt{3}/2 \end{pmatrix}$$

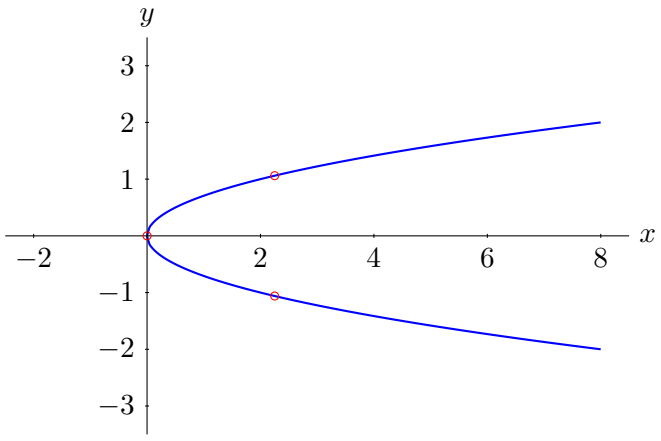
We see that the determinant of this matrix is $15 \cdot 3/4 - 9/4 = 9$. Hence, $(x,y) = (1/2, \sqrt{3}/2)$ is a local minimum point for f. If we insert the second point, we get

$$H(f)(1/2, -\sqrt{3}/2) = \begin{pmatrix} -5\sqrt{3}/2 & 3/2 \\ 3/2 & -3\sqrt{3}/2 \end{pmatrix}$$

The determinant becomes 9, and hence $(x,y) = (1/2, -\sqrt{3}/2)$ is a local maximum point for f.

(d) If we for instance let $y = x$, the function f is given by $f(x,x) = 4x^3 - 4x^2$. We see that $f(x,x) \to \infty$ when $x \to \infty$ and $f(x,x) \to -\infty$ when $x \to -\infty$. This means that f neither has a global maximum- or minimum value.

Problem 5.



(a) Since the constraint can be written as $x - 2y^2 = 0$ in the standard form, we use the Lagrange multiplier method with $\mathcal{L} = y^2 - x^2 + 4x - 4 - \lambda(x - 2y^2)$ in order to find candidate points. The Lagrange conditions are

$$\mathcal{L}'_x = -2x + 4 - \lambda = 0$$
$$\mathcal{L}'_y = 2y - \lambda(-4y) = 0$$
$$x - 2y^2 = 0$$

The second equation gives $2y(1 + 2\lambda) = 0$, and this implies that $y = 0$ or $\lambda = -1/2$. If $y = 0$, the constraint implies that $x = 0$, and the first equation that $\lambda = 4$. If $\lambda = -1/2$, the first

equation implies that $-2x + 9/2 = 0$, or that $x = 9/4$. If we insert this into the constraint, we get $9/4 = 2y^2$, which gives $y^2 = 9/8$, or $y = \pm 3/\sqrt{8}$. Hence, we find the following three candidate points from the Lagrange multiplier method:

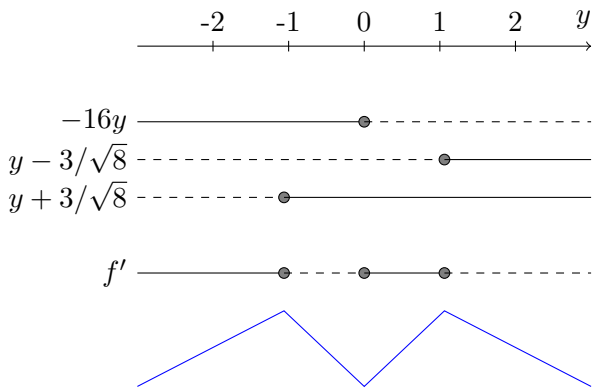
$$(x,y;\lambda) = (0,0;4), (9/4, \pm 3/\sqrt{8}; -1/2)$$

(b) We can solve the equation $x = 2y^2$ for y, which gives $y^2 = x/2$, or $y = \pm\sqrt{x/2}$ when $x \geq 0$. Therefore, the curve given by $x = 2y^2$ consists of the graphs of the two functions $\pm\sqrt{x/2}$ with domain $x \geq 0$. These are shown in the figure above with the three points from (a) marked in red.

(c) The simplest way to solve the problem is to substitute $x = 2y^2$ into the expression for f. Then, we find the function

$$f(2y^2,y) = y^2 - (2y^2)^2 + 4(2y^2) - 4 = -4y^4 + 9y^2 - 4$$

This is a function in one variable with derivative $-16y^3+18y = -16y(y^2-9/8)$. By setting the derivative equal to zero, we get $y = 0$ or $y^2 = 9/8$, which gives $y = \pm 3/\sqrt{8} \approx \pm 1.06$. Hence, the stationary points are the same as the ones we found in (a). We make a sign diagram for the derivative as shown below. Since $y = \pm 3/\sqrt{8}$ gives $f = -4(9/8)^2+9(9/8)-4 = 17/16$, the



points $(9/4, \pm 3/\sqrt{8}; -1/2)$ are maximum points, with maximum value $f_{\max} = 17/16$. We also see from the sign diagram that the problem does not have a minimum value since $y \to \pm\infty$ gives $f \to -\infty$.

Alternatively, it is possible to consider the level curves $f(x,y) = c$ and draw these into the figure from (b), see the figure below. For $c = 0$, we get $y = \pm(x-2)$, which is shown as the dashed straight lines (asymptotes). For $c < 0$, we get curves to the right and to the left of the asymptotes, such as $f(x,y) = -4$ which is shown in the figure passing through (0,0). Since these curves will intersect $x = 2y^2$ (to the right) when $c \to -\infty$, the problem does not have a minumum. For $c > 0$, we get curves above and below the asymptotes, such as $f(x,y) = 17/16$ which is shown in the figure and which passes through the points $(9/4, \pm 3/\sqrt{8})$. Since this level curve is tangent to $x = 2y^2$, there will be no intersection points if $c > 17/16$, and hence $f_{\max} = 17/16$ in the two points where the level curve is tangent to $x = 2y^2$.