EVALUATION GUIDELINES - Course paper

EBA 29101
Mathematics for Business Analytics

Department of Economics

Start date:	03.10.2019	Time 09:00
Finish date:	11.10.2019	Time 12:00

For more information about formalities, see examination paper.

Term paper - EBA2911[1] Mathematics for Business Analytics

3 Oct. – 11 Oct. 2019

SOLUTIONS

Problem 1

a) We put $v = x^2$ and get the quadratic equation $v^2 - 5v - 36 = 0$ with solutions $v = 9$, $v = -4$, that is $x^2 = 9$, $x^2 = -4$. Because $x^2 \geqslant 0$ we only get $\underline{\underline{x = \pm 3}}$.

b) We put $v = \sqrt{x}$ and get the same equation $v^2 - 5v - 36 = 0$ as in (a). Hence we get $\sqrt{x} = 9$ or $\sqrt{x} = -4$. Because $\sqrt{x} \geqslant 0$ we only get $\underline{\underline{x = 81}}$.

c) We put $v = \frac{1}{x}$ and again get the same equation $v^2 - 5v - 36 = 0$ as in (a). Hence we get $\frac{1}{x} = 9$ or $\frac{1}{x} = -4$, which gives $\underline{\underline{x = \frac{1}{9}}}$, $\underline{\underline{x = -\frac{1}{4}}}$.

d) We isolate one of the roots

$$\sqrt{2x-1} = 5 - \sqrt{x-1}$$

and square each side

$$2x - 1 = 5^2 - 10\sqrt{x-1} + (x-1)$$

Then we isolate the remaining root

$$x - 25 = -10\sqrt{x-1}$$

and square each side

$$x^2 - 50x + 625 = 100(x-1)$$

that is

$$x^2 - 150x + 725 = 0$$

which has solutions $x = 5$, $x = 145$. We test these solutions to see if they are solutions of the original equation.

For $x = 145$ the left hand side $\sqrt{2 \cdot 145 - 1} + \sqrt{145 - 1} = 19 + 12 = 31$ which is not equal to the right hand side.

For $x = 5$ the left hand side $\sqrt{2 \cdot 5 - 1} + \sqrt{5 - 1} = 3 + 2 = 5$ which also is the right hand side. The conclusion is that $\underline{\underline{x = 5}}$ is the only solution.

Problem 2

a) The inequality has 0 on the right hand side and a factorised fraction on the the left hand side. Then we can use a sign diagram to solve the inequality:

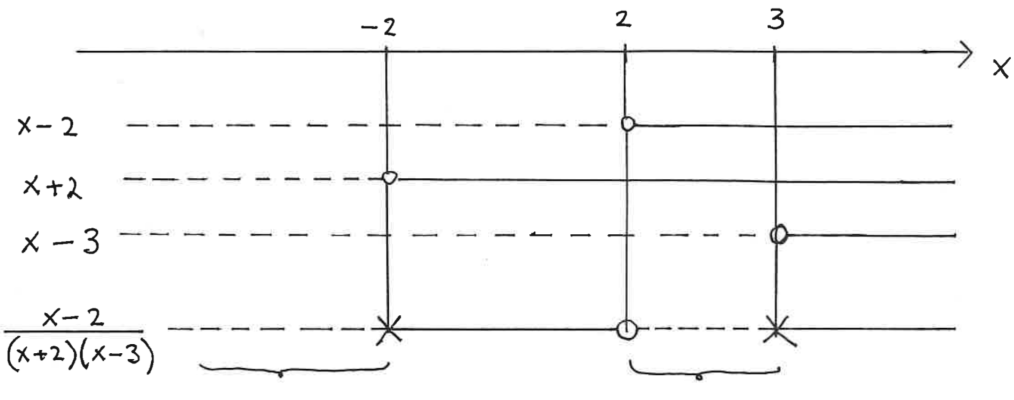

Figure 1: Sign diagram in 2a

[1]Exam code EBA29101

which gives $\underline{\underline{x < -2 \text{ or } 2 \leqslant x < 3}}$. Alternative way of writing: $\underline{\underline{x \in \langle \leftarrow, -2\rangle \cup [2,3\rangle}}$.

b) We rewrite to an equivalent inequality with 0 on the right hand side and one fraction on the left hand side:

$$\frac{(x-2)+(x+2)(x-3)}{(x+2)(x-3)} \leqslant 0$$

Then we resolve the paranteses, collect terms and factorise in the numerator:

$$\frac{(x-\sqrt{8})(x+\sqrt{8})}{(x+2)(x-3)} \leqslant 0$$

Now we can use a sign diagram:

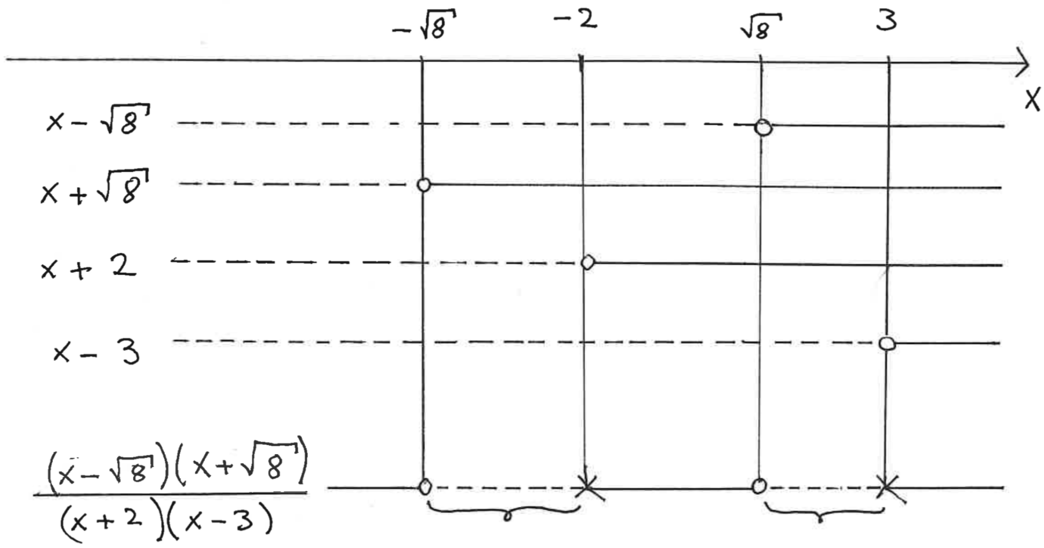

Figure 2: Sign diagram in 2b

which gives $\underline{\underline{-\sqrt{8} \leqslant x < -2 \text{ or } \sqrt{8} \leqslant x < 3}}$. Alternative way of writing: $\underline{\underline{x \in [-\sqrt{8}, -2\rangle \cup [\sqrt{8}, 3\rangle}}$.

Problem 3

We have

$$0.1x^4 - 2.4x^3 + 11.8x^2 + 31.2x + 16.9 = 0.1(x^4 - 24x^3 + 118x^2 + 312x + 169)$$

We insert $x = -1$ and get

$$(-1)^4 - 24(-1)^3 + 118(-1)^2 + 312(-1) + 169 = 1 + 24 + 118 - 312 + 169 = 0$$

Hence $(x+1)$ is a factor. Use polynomial division to find

$$\begin{array}{l}
\big(x^4 - 24x^3 + 118x^2 + 312x + 169\big) : \big(x+1\big) = x^3 - 25x^2 + 143x + 169 \\
\underline{-x^4 \quad -x^3} \\
\qquad -25x^3 + 118x^2 \\
\qquad \underline{25x^3 \quad +25x^2} \\
\qquad\qquad 143x^2 + 312x \\
\qquad\qquad \underline{-143x^2 - 143x} \\
\qquad\qquad\qquad 169x + 169 \\
\qquad\qquad\qquad \underline{-169x - 169} \\
\qquad\qquad\qquad\qquad 0
\end{array}$$

We insert $x = 13$ in $x^3 - 25x^2 + 143x + 169$ and get

$$13^3 - 25 \cdot 13^2 + 143 \cdot 13 + 169 = 2197 - 4225 + 1859 + 169 = 0$$

Hence $(x - 13)$ is a factor. Use polynomial division to find

$$\begin{array}{l}
\big(\quad x^3 - 25x^2 + 143x + 169\big) : \big(x - 13\big) = x^2 - 12x - 13 \\
\underline{-x^3 + 13x^2} \\
\qquad -12x^2 + 143x \\
\qquad \underline{\ \ 12x^2 - 156x} \\
\qquad\qquad -13x + 169 \\
\qquad\qquad \underline{\ \ 13x - 169} \\
\qquad\qquad\qquad 0
\end{array}$$

We find that $x^2 - 12x - 13$ has the zeros $x = -1$ and $x = 13$, which gives
$x^2 - 12x - 13 = (x + 1)(x - 13)$. Then
$0.1x^4 - 2.4x^3 + 11.8x^2 + 31.2x + 16.9 = \underline{\underline{0.1(x + 1)^2(x - 13)^2}}$.

Problem 4

a) If the principal is K_0, the balance 10 years from now will be $K_0 \cdot 1.021^{10}$ which is supposed to be 2 million. We therefore solve the equation $K_0 \cdot 1.021^{10} = 2$ mill and get the present value $K_0 = 2 \cdot 1.021^{-10}$ mill $= \underline{\underline{1\,624\,697.73}}$.

b) After 6 years the balance is $1\,624\,697.73 \cdot 1.021^6 = 1\,840\,462.73$. Deposited 4 years in an account with 2.7% interest gives $1\,840\,462.73 \cdot 1.027^4 = \underline{\underline{2\,047\,428.77}}$.

c) We use the expressions instead of the intermediate values:

$$\begin{aligned}
2\,047\,428.77 &= 1\,840\,462.73 \cdot 1.027^4 = 1\,624\,697.73 \cdot 1.021^6 \cdot 1.027^4 \\
&= 2 \text{ mill} \cdot 1.021^{-10} \cdot 1.021^6 \cdot 1.027^4 \\
&= 2 \text{ mill} \cdot 1.021^{-4} \cdot 1.027^4 \\
&= 2\,\text{mill} \cdot \left(\frac{1.027}{1.021}\right)^4
\end{aligned}$$

d) If the principal is K_0 the balance after 10 years will be $K_0 \cdot 1.021^6 \cdot 1.027^4 = K_0 \cdot 1.260\,191$ which gives the equation $K_0 \cdot 1.260\,191 = 3$ mill and $K_0 = 3$ mill : $1.260\,191 = \underline{\underline{2\,380\,592.32}}$.

e) We use the expressions instead of the intermediate values in (c):

$$1\,189\,532.60 = \frac{3 \text{ mill}}{1.260\,191} = \frac{3 \text{ mill}}{1.027^4 \cdot 1.021^6}$$

Problem 5

a) The future value 6 years from now is:

$$K_6 = -20 \cdot 1.1^6 - 20 \cdot 1.1^5 + 30 \cdot 1.1 + 45 = \underline{\underline{10.36}}$$

b) The present value is:

$$K_0 = -20 - 20 \cdot 1.1^{-1} + 30 \cdot 1.1^{-5} + 45 \cdot 1.1^{-6} = \underline{\underline{5.85}}$$

c) We have $5.85 \cdot 1.1^6 = 10.36$.

If A is a payment n years from now the presnt value is $A \cdot (1+r)^{-n}$ while the future value 6 years from now is $A \cdot (1+r)^{6-n}$. To get from the present value $A \cdot (1+r)^{-n}$ to the future value 6 years from now we multiply with $(1+r)^6$. This number is hence the same whenever the payment happens. When we multiply the sum of the present values of the many payments at different times by $(1+r)^6$ each of the present values will be multiplied by $(1+r)^6$ (we 'multiply into the paranthesis'). This gives the sum of the future values of each of the payments which precisely is the future value of the cash flow.

Problem 6

a) The period rate is 6% : $12 = 0.5\%$ and it is $25 \cdot 12 = 300$ periods. First payment is $5 \cdot 12 = 60$ periods from now. The geometric series is hence

$$\underline{\underline{\frac{15\,000}{1.005^{60}} + \frac{15\,000}{1.005^{61}} + \cdots + \frac{15\,000}{1.005^{358}} + \frac{15\,000}{1.005^{359}}}}$$

If we sum this geometric series 'from the right' such that $a_1 = \frac{15\,000}{1.005^{359}}$, $k = 1.005$ and $n = 300$ we get

$$\frac{15\,000}{1.005^{359}} \cdot \frac{1.005^{300} - 1}{0.005} = \underline{\underline{1\,734\,620.76}}$$

b) The growth factor for one period is $e^{0.005}$. Apart from that it is as in (a). Hence we get the geometric series of present values:

$$\underline{\underline{\frac{15\,000}{e^{60\cdot 0.005}} + \frac{15\,000}{e^{61\cdot 0.005}} + \cdots + \frac{15\,000}{e^{358\cdot 0.005}} + \frac{15\,000}{e^{359\cdot 0.005}}}}$$

We calculate the sum of this geometric series 'from the right' such that $a_1 = \frac{15\,000}{e^{359\cdot 0.005}}$, $k = e^{0.005}$ and $n = 300$ and we get

$$\frac{15\,000}{e^{359\cdot 0.005}} \cdot \frac{e^{300\cdot 0.005} - 1}{e^{0.005} - 1} = \underline{\underline{1\,730\,877.99}}$$

Problem 7

a) The cash flow of payments is as follow (n years):

Year	15	16	17	...	$n+14$
Payment	A	$A \cdot 1.03$	$A \cdot 1.03^2$	...	$A \cdot 1.03^{n-1}$

This gives the following sum of present values (with $n = 25$):

$$\underline{\underline{\frac{A}{1.05^{15}} + \frac{A \cdot 1.03}{1.05^{16}} + \frac{A \cdot 1.03^2}{1.05^{17}} + \cdots + \frac{A \cdot 1.03^{24}}{1.05^{39}}}}$$

If we read the series 'backwards' we get the geometric series with $\underline{\underline{a_1 = \frac{A \cdot 1.03^{24}}{1.05^{39}}}}$, $\underline{\underline{k = \frac{1.05}{1.03}}}$ and $n = 25$. The sum of the series is then given as

$$\frac{A \cdot 1.03^{24}}{1.05^{39}} \cdot \frac{(\frac{1.05}{1.03})^{25} - 1}{\frac{1.05}{1.03} - 1} = A \cdot \frac{1.03^{24} \cdot \frac{1.05^{25}}{1.03^{25}} - 1.03^{24}}{1.05^{39} \cdot \frac{1.05 - 1.03}{1.03}} = A \cdot \frac{\frac{1.05^{25}}{1.03} - 1.03^{24}}{1.05^{39} \cdot \frac{0.02}{1.03}}$$
$$= A \cdot \frac{1.05^{25} - 1.03^{25}}{1.05^{39} \cdot 0.02}$$

The present value is given as 20 million so we get the equation

$$A \cdot \frac{1.05^{25} - 1.03^{25}}{1.05^{39} \cdot 0.02} = 20\,000\,000$$

which gives

$$A = 20\,000\,000 \cdot \frac{1.05^{39} \cdot 0.02}{1.05^{25} - 1.03^{25}} = \underline{\underline{2\,074\,847.72}}$$

b) The sum of the present values continues the series in (a):

$$\frac{A}{1.05^{15}} + \frac{A \cdot 1.03}{1.05^{16}} + \frac{A \cdot 1.03^2}{1.05^{17}} + \cdots + \frac{A \cdot 1.03^{n-1}}{1.05^{n+14}} + \ldots$$

This is an infinite geometric series with $a_1 = \frac{A}{1.05^{15}}$ and $k = \frac{1.03}{1.05}$ (which is less that 1). Then the sum is (by a formula in the textbook)

$$a_1 \cdot \frac{1}{1-k} = \frac{A}{1.05^{15}} \cdot \frac{1}{1 - \frac{1.03}{1.05}} = \frac{A}{1.05^{15}} \cdot \frac{1.05}{0.02} = \frac{A}{1.05^{14} \cdot 0.02}$$

The present value is given as 20 million so we get the equation

$$\frac{A}{1.05^{14} \cdot 0.02} = 20\,000\,000$$

which gives

$$A = 20\,000\,000 \cdot 1.05^{14} \cdot 0.02 = \underline{\underline{791\,972.64}}$$

Problem 8

a) From the graph we see that the symmetry-axis is $x = 20$ and the minimal value is $y = 10$. That gives the standard form $f(x) = a(x-20)^2 + 10$. We also see that the point $(25, 15)$ is on the graph. This gives the equation $a(25-20)^2 + 10 = 15$, that is $a = \frac{15-10}{25} = 0.2$. Hence $f(x) = \underline{\underline{0.2(x-20)^2 + 10}}$.

b) From the graph we see that the centre of the ellipse is $(4, 3)$ with half-axes $a = 10 - 4 = 6$ and $b = 6 - 3 = 3$. This give the standard equation

$$\underline{\underline{\frac{(x-4)^2}{36} + \frac{(y-3)^2}{9} = 1}}$$

Problem 9

a) We se that the points $A = (2, 6)$, $B = (6, 10)$, $C = (12, 4)$ and $D = (8, 0)$ is on the graph. We see that $ABCD$ is a rectangle (the edges have slope ± 1). Then the intersection of the diagonals is equidistant from each of the branches of the hyperbola. The intersection point has coordinates $(7, 5)$. The horizontal and vertical asymptote cross each other in this point. Hence the vertical asymptote is the line $x = 7$ and the horizontal asymptote the line $y = 5$. Then the expression for the hyperbola is given by the standard form $f(x) = 5 + \frac{a}{x-7}$. We know that $f(8) = 0$, that is $5 + \frac{a}{8-7} = 0$ which gives $a = -5$. Hence $f(x) = \underline{\underline{5 - \frac{5}{(x-7)}}}$.

b) When x becomes big positive or big negative, the number $\frac{5}{(x-7)}$ approaches 0 (from above and below, respectively). Hence $5 - \frac{5}{(x-7)}$ approaches 5 (from below and from above, respectively). Hence the line $y = 5$ is the horizontal asymptote for $f(x)$ which agrees with what we found from the graph. We also have

$$\frac{5}{(x-7)} \xrightarrow[x \to 7^+]{} \infty \quad \text{og} \quad \frac{5}{(x-7)} \xrightarrow[x \to 7^-]{} -\infty$$

which gives

$$5 - \frac{5}{(x-7)} \xrightarrow[x\to 7^+]{} -\infty \quad \text{og} \quad 5 - \frac{5}{(x-7)} \xrightarrow[x\to 7^-]{} \infty$$

Hence the line $x = 7$ is a vertical asymptote for $f(x)$ which also agrees with what we found from the graph. Since the function $f(x)$ has a defined value for all $x \neq 7$ this is the only vertical asymptote.

Problem 10

a) The expression is

$$3\left(x-(5-\sqrt{3})\right)\left(x-(5+\sqrt{3})\right) = 3\left(x^2-10x+(5-\sqrt{3})(5+\sqrt{3})\right) = 3(x^2-10x+25-3)$$
$$= \underline{\underline{3x^2-30x+66}}$$

b) Here we have a double root. The expression is

$$3(x-(-11))(x-(-11)) = 3(x+11)^2 = \underline{\underline{3x^2+66x+363}}$$

Problem 11

Let $x = t$ be the smallest root. Then $x = t+3$ is the biggest root. There are two cases. Either t is a double root (case A) or $t+3$ is a double root (case B).

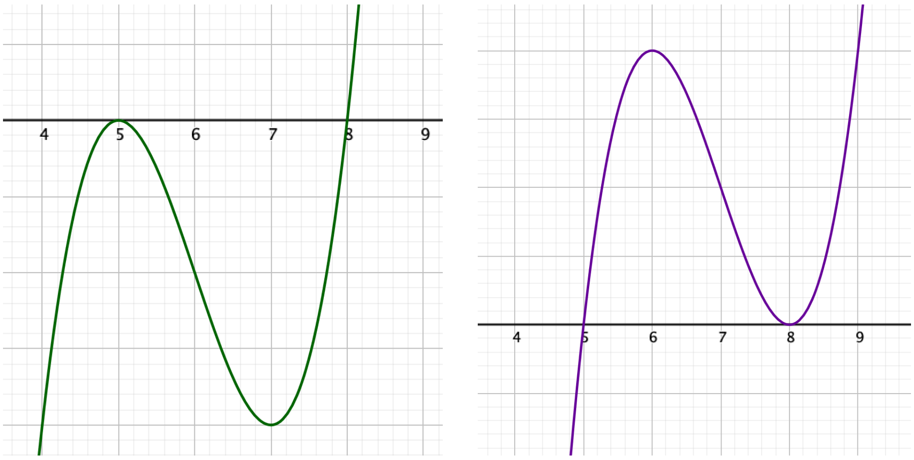

Figure 3: Case A and B with $t = 5$

In case A we get the third degree expression

$$(x-t)^2(x-t-3) = x^3-3(t+1)x^2+3t(t+2)x-t^2(t+3)$$

In case B we get the third degree expression

$$(x-t)(x-t-3)^2 = x^3-3(t+2)x^2+3(t^2+4t+3)x-t(t^2+6t+9)$$

We can write this as:

$$\underline{\underline{\begin{cases} x^3-3(t+1)x^2+3t(t+2)x-t^2(t+3) & \text{in case A} \\ x^3-3(t+2)x^2+3(t^2+4t+3)x-t(t^2+6t+9) & \text{in case B} \end{cases}}}$$